Lecture 17 The many Forms of the Navier Stokes Equation

1. The Navier-Stokes Equation

Last time we saw how conservation laws in general can be formulated, and we derived the Cauchy conservation of momentum equation and illustrated its use (on sprinklers among other things).

Cauchy's conservation of momentum equation

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \qquad (5\text{-}4)$$

Now we want to derive the Navier Stokes equation (and its variants conservation / continuity, the Euler equation (no viscosity), and the Bernoulli equation). We want our equations in terms of u_i alone and so we need a constitutive relation to relate τ_{ij} to $u_{i,j}$.

At rest $\tau_{ij} = -p\delta_{ij}$. At rest only normal components of stress can act on a fluid. Since by definition tension is positive, the normal components must be negative. Furthermore they must be the same in all directions, so, at rest: $\tau_{ij} = -p\,\delta_{ij}$, where p is the thermodynamic pressure.

For a moving fluid an additional stress comes into play because of viscous stress. Call it σ_{ij}. So for a moving fluid

$$\tau_{ij} = -p\,\delta_{ij} + \sigma_{ij}.$$

In a linear fluid σ_{ij} is related to the deformation tensor, $\underline{\underline{e}}$:

$$\sigma_{ij} = K_{ijmn}\,e_{mn}$$

If the medium is isotropic and the stress tensor symmetric,

(5-5) Linear Newtonian fluid

$$\boxed{\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm}\delta_{ij}}.$$

Substituting $\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm}\delta_{ij}$

into

$$\tau_{ij} = p_n \delta_{ij} + \sigma_{ij}$$

with Stokes assumption that $\lambda + \frac{2}{3}\mu = 0$,

given (as we will show in next page):

(5-6) $$\tau_{ij} = -\left\{ p_n + \frac{2}{3}\mu \nabla\cdot u \right\}\delta_{ij} + 2\mu e_{ij}$$

Simplify + substitute into Cauchy equation:

Simplify $\partial_j (2\mu e_{ij})$: $2\mu\, \partial_j \left(e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)\right)$

$= \mu\left(\partial^2 u_i + \partial_i \nabla\cdot u\right)$

$\mu\nabla^2 \underline{u} + \mu \underline{\nabla}\,\underline{\nabla}\cdot\underline{u}$

Cauchy eqn
$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

yields:

(5-8) Navier stokes equation (bound - incompressible)
$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p_n + \rho \underline{g} + \mu \nabla^2 \underline{u} + \frac{1}{3}\mu \underline{\nabla}\,\underline{\nabla}\cdot\underline{u}$$

if flow is far from boundaries, viscous effect → 0

and

Euler equation (5-9)
$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{g}$$

Now lets look at details:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm}\delta_{ij}$$

$$\tau_{ij} = -p_{th}\delta_{ij} + \sigma_{ij}$$

(5-10) $$\tau_{ij} = -p_{th}\delta_{ij} + 2\mu e_{ij} + \lambda e_{mm}\delta_{ij}$$

(a) multiply by δ_{ij} to contract

$$\delta_{ij}\tau_{ij} = \tau_{ii} = -p_{th}\delta_{ij}\delta_{ij} + 2\mu e_{ii} + \lambda e_{mm}\delta_{ij}\delta_{ij}$$

$\delta_{ij}\delta_{ij} = 3$ — from homework — $\delta_{ij}\delta_{ij} = 3$

$$\tau_{ii} = -3p_{th} + (2\mu + 3\lambda)e_{mm}$$

$$p_{th} = \frac{-1}{3}\tau_{ii} + \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot u$$

(5-11) $$\bar{p} \equiv -\tfrac{1}{3}\tau_{ij} \quad \text{mean pressure}$$

$$p_{th} - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right)\underline{\nabla}\cdot\underline{u} = K\,\underline{\nabla}\cdot\underline{u}$$

coeff of bulk viscosity

If incompressible or if $K = 0$, $p_{th} = \bar{p}$ (Stokes assumption)

Taking $K = 0$, $\lambda = -\frac{2}{3}\mu$ and from (5-10):

(5-12) $$\boxed{\tau_{ij} = -\left(p_{th} + \frac{2}{3}\mu\nabla\cdot u\right)\delta_{ij} + 2\mu e_{ij}}$$

which is constitutive relation for a Newtonian fluid

2 The Vorticity Equation

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$$ Incompressible Navier-Stokes

$\underline{\omega} = \nabla \times \underline{u}$ Vorticity

$\underline{g} \equiv -\nabla \Pi$

so $$\nabla \times \left(\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \Pi + \frac{\mu}{\rho}\nabla^2 \underline{u} \right)$$

assuming ρ = constant, $\nu = \mu/\rho$:

Vorticity diffuses like Temperature!

$$\frac{D\underline{\omega}}{Dt} = \nu \nabla^2 \omega \sim \frac{DT}{Dt} = \kappa \nabla^2 T$$

This is NOT QUITE RIGHT because curl operator is a spatial, not material, coordinate. Doing it correctly (see flow #2)

(e.g. $\nabla \cdot \frac{D\underline{u}}{Dt} \neq \frac{D\underline{\omega}}{Dt}$)

given

(5-13) $$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$$ Vorticity equation

↑ change due to stretching & tilting of vortex lines

↑ Diffusion of vorticity

note: $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$
$\omega = \nabla \times u = A$
so $\nabla \cdot A = 0$

Need to do operations in Eulerian coordinates to get extra term in vorticity equation.

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$

$$\nabla \cdot \omega = 0 = \underline{\nabla} \cdot \underline{\nabla} \times \underline{u} \equiv 0$$

$$\nabla \times \left\{ \frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{1}{\rho} \nabla p - \nabla \Pi + \nu \nabla^2 u \right\}$$

$$\partial_j \partial_j u_i = u_j (\partial_j u_i - \partial_i u_j) + u_j \partial_i u_j$$

$$= -u_j (\partial_i u_j - \partial_j u_i) + u_j \partial_i u_j$$

$$= -u_j \epsilon_{ijk} \omega_k \quad + \frac{1}{2} \partial_i u_j u_j$$

Lemma

$$u \times \nabla \times u = \epsilon_{ijk} u_j \epsilon_{klm} \partial_l u_m$$

$$= \epsilon_{ijk} \epsilon_{klm} u_j \partial_l u_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) u_j \partial_l u_m$$

$$= u_j \partial_i u_j - u_j \partial_j u_i$$

$$= u_j (\partial_i u_j - \partial_j u_i)$$

$$\frac{\partial \omega_n}{\partial t} + \epsilon_{nqi} \partial_q \left\{ -u_j \epsilon_{ijk} \omega_k + \frac{1}{2} \partial_i (u_j u_j) \right\}$$

$$- \epsilon_{nqi} \epsilon_{ijk} \partial_q u_j \omega_k \quad + \underbrace{\frac{1}{2} \epsilon_{nqi}}_{\text{antisymm}} \underbrace{\partial_q \partial_i}_{\text{symm}} u_j^2$$

$$- (\delta_{nj} \delta_{qk} - \delta_{nk} \delta_{qj}) \partial_q u_j \omega_k \qquad = 0$$

$$\frac{\partial w}{\partial t} + \left\{ \begin{array}{l} -\partial_k (u_n w_k) + \partial_j (u_j w_n) \\ -u_n \partial_k w_k - w_k \partial_k u_n + w_n \partial_j u_j + u_j \partial_j w_n \\ \quad \nabla \cdot w_k = 0 \qquad\qquad\qquad\qquad \nabla \cdot u = 0 \\ +u_j \partial_j w_n - w_k \partial_k u_n \end{array} \right.$$

(5-13)
$$\boxed{\frac{Dw}{Dt} - \underline{w} \cdot \nabla u = \nu \nabla^2 \underline{w}}$$

QED

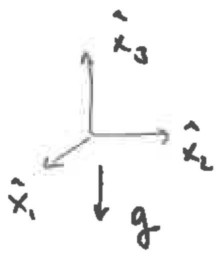
Message – Be careful!

(Think for yourself)

3. The Bernoulli Equation

Go back to the inviscid form of momentum conservation (e.g. the Euler equation):

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{g}$$



$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -\partial_i (gz) - \frac{1}{\rho} \partial_i p$$

$$u_j \left(\partial_j u_i - \partial_i u_j \right) + u_j \partial_i u_j$$

$$\parallel \qquad\qquad \parallel$$

Antisymmetric rotation tensor

$$u_j r_{ij} \quad + \quad \partial_i \left(\tfrac{1}{2} u_j u_j \right)$$

$$\parallel$$

$$u_j \epsilon_{ijk} \omega_k \quad + \quad \partial_i \left(\tfrac{1}{2} q^2 \right)$$

$$\parallel$$

$$-\underline{u} \times \underline{\omega}$$

Then

$$\frac{\partial u_i}{\partial t} + \partial_i \left(\tfrac{1}{2} q^2 \right) + \frac{1}{\rho} \partial_i p + \partial_i g z = (u \times \omega)_i$$

Now assume $\rho = \rho(p)$ which is barotropic flow

Then we can show (and will below):

$$\frac{1}{\rho} \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i} \int \frac{dp}{\rho}$$

PERFECT DIFFERENTIAL - Depends only on end points!

So we can write:

$$\frac{du_i}{dt} + \frac{\partial}{\partial x_i}\underbrace{\left[\frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz\right]}_{\equiv B} = (\underline{u}\times\underline{w})_i$$

Bernoulli's equation (5-14)

$$\frac{\partial \underline{u}}{\partial t} + \nabla \underline{B} = \underline{u}\times\underline{w}$$

Bernoulli's function

$$B = \frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz$$

(a) For steady flow $\frac{\partial u}{\partial t} = 0$ and:

$$\underbrace{\underline{\nabla} B}_{\text{vector} \perp \text{ to } B = \text{constant}} = \underbrace{\underline{u}\times\underline{w}}_{\text{vector} \perp \text{ to both } \underline{u} \text{ and } \underline{w}}$$

∴ B must be constant along stream lines + vortex lines

(5-15)

$$\frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz = \text{constant along streamlines + vortex lines}$$

(b) for unsteady irrotational flow

$\nabla\phi = \nabla \times u$

If irrotational, $u = \nabla\phi$ (because $\underline{u} \times \underline{w} = 0$), and

$$\nabla\left(\frac{\partial\phi}{\partial t} + B\right) = 0$$

$\nabla(u \times w) = 0$ — grad curl $= 0$

gradient is constant ∴ can vary only with t, $\frac{\partial\phi}{\partial t} + B$ independ of location

(5-16)
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz = F(t)$$
indep of location

a lemma + then some applications: ....

Lemma: $\int\frac{dp}{\rho}$ = ~~perfect differential~~ if $\rho = \rho(p)$ only (Barotropic flow)

$$\int\frac{dp}{\rho} = \int\frac{1}{\rho}\frac{dp}{d\rho}\,d\rho = \int\frac{dP}{d\rho}\,d\rho = \int dP$$

$\frac{dP}{\rho}$

which is perfect differential.

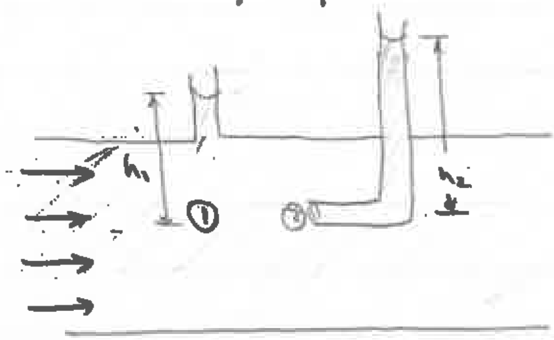
$$\frac{dP}{d\rho} = \frac{1}{\rho}\frac{dp}{d\rho}$$

a function of ρ alone

c. Bernoulli Equation Example

(1.) Pitot Tube

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} + gz = \text{constant}$$



② start $p = p_2$

$$\int_{p_1}^{p_1} \frac{dp}{\rho} + \frac{1}{2}u_1^2 = \int_{p_1}^{p_2} \frac{dp}{\rho} + \frac{1}{2}u_2^2$$

$$\frac{1}{2}u_1^2 = \frac{p_2 - p_1}{\rho}$$

① ②

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho} + \frac{u_2^2}{2} = \frac{p_2}{\rho}$$

($u_2^2 \rightarrow 0$)

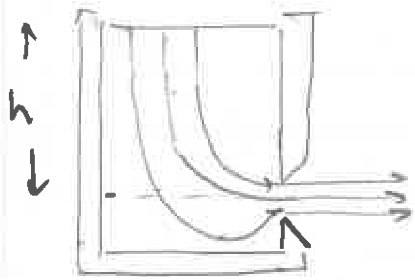
$$\therefore u_1 = \sqrt{2(p_2 - p_1)/\rho}$$

$$p_1 = \rho g h_1 \quad , \quad p_2 = \rho g h_2$$

$$\therefore \boxed{u_1 = \sqrt{2g(h_2 - h_1)}}$$

Can measure fluid velocity very easily!

(2.) Orifice in a Tank



$$B = \text{constant along streamline}$$

$$= \frac{q^2}{2} + \frac{p}{\rho} + gz$$

$$= \frac{p_{atm}}{\rho} + gh \quad \text{at top}$$

$$= \frac{p_{atm}}{\rho} + \frac{u^2}{2} \quad \text{at jet}$$

(because can use $p_{atm} = 0$ as reference)

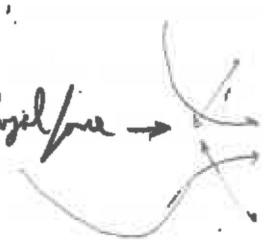
$$\therefore \quad u = \sqrt{2gh}$$

$$\dot{m} = \text{mass flux out} = \rho A_c u$$

$$\boxed{\dot{m} = \rho A_c \sqrt{2gh}}$$

note centrifugal force of converging streamlines

causes $A_c^{eff} \simeq 0.62\, A_c$.



4.1 The Energy Bernoulli Equation

If Steady state, no heat conduction, no viscous stresses;

then $\tau_{ij} = -p\delta_{ij}$ and, expanding $\frac{D}{Dt}$ (5-16) becomes

(energy eqn for cons. law written $\rho\delta$)

(advection part)

$$\rho u_i \frac{\partial}{\partial x_i}\left(e + \frac{q^2}{2} + gz\right) = -\frac{\partial}{\partial x_i}(u_i p)$$

Since steady state mass conservation requires, o

$$\frac{\partial(\rho u_i)}{\partial x_i} = 0 \quad , \quad -\frac{\partial}{\partial x_i}\left(\rho u_i \frac{p}{\rho}\right) = -\frac{p}{\rho}\frac{\partial(\rho u_i)}{\partial x_i} + \rho u_i \frac{\partial}{\partial x_i}\frac{p}{\rho},$$

and

$$\rho u_i \frac{\partial}{\partial x_i}\left(e + \frac{p}{\rho} + \frac{q^2}{2} + gz\right) = 0 \qquad (5\text{-}18)$$

" constant along streamline, because $u \cdot \nabla() = 0$

now $h = e + \frac{p}{\rho}$, where h is enthalpy. Thus

$\nabla\left(h + \frac{q^2}{2} + gz\right)$ must be perpendicular to $\underline{u}$, and

$h + \frac{q^2}{2} + gz$ is therefore constant along a streamline

This is useful in high speed flows to show how kinetic energy, enthalpy and potential energy inter-relate.

Lemma

This can be shown using the epsilon-delta relationship:

$$\nabla \times \nabla \times u = \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l u_m$$

$$= \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l u_m$$

but $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l u_m$$

$$= \partial_j \partial_i u_j - \partial_j \partial_j u_i$$

$$= \nabla(\nabla \cdot u) - \nabla^2 u$$

0

While we're at it, note:

$$u \times \nabla \times u = \epsilon_{ijk} u_j \epsilon_{klm} \partial_l u_m$$

$$= \epsilon_{ijk} \epsilon_{klm} u_j \partial_l u_m$$

$$(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) u_j \partial_l u_m = u_j \partial_i u_j - u_j \partial_j u_i$$

$$= u_j (\partial_i u_j - \partial_j u_i)$$

Substituting (5-6) into the conservation of momentum equation (5-4) results in:

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\left(-\left(p + \frac{2}{3}\mu \nabla\cdot u\right)\delta_{ij} + 2\mu e_{ij}\right)$$

(5-7a) Navier-Stokes Equation

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left(2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\delta_{ij}\right)$$

If temperature gradients in fluid are not too large, $\mu(T)$ can be considered constant, and

$$2\mu \frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mu\left(\frac{\partial u_i}{\partial x_j \partial x_j} + \frac{\partial u_j}{\partial x_i \partial x_j}\right)$$

$$= \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i}\nabla\cdot \underline{u}$$

$= -\mu(\nabla\times\omega)$ see book p. 98

(5-7b)

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\nabla^2 u_i + \frac{\mu}{3}\left(\frac{\partial}{\partial x_i}\nabla\cdot \underline{u}\right)$$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i - \mu(\nabla\times\omega) - \frac{4}{3}\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)$$

1. If $\nabla\cdot u = 0$ (incompressible)

(5-8) Incompressible Navier-Stokes

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu\nabla^2 \underline{u}$$

2. If far enough from boundaries the viscous effect $\to 0$

(5-9) Euler Equation

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g}$$

Plan #3 L7

However, if the temperature differences are small within the fluid, then μ can be taken outside the derivative in equation (4.44), which then reduces to

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + 2\mu \frac{\partial e_{ij}}{\partial x_j} - \frac{2\mu}{3} \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u})$$
$$= -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \left[\nabla^2 u_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u})\right],$$

where

$$\nabla^2 u_i \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2},$$

is the Laplacian of u_i. For incompressible fluids $\nabla \cdot \mathbf{u} = 0$, and using vector notation, the Navier–Stokes equation reduces to

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}.} \quad \text{(incompressible)} \tag{4.45}$$

If viscous effects are negligible, which is generally found to be true far from boundaries of the flow field, we obtain the *Euler equation*

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}. \tag{4.46}$$

Comments on the Viscous Term

For an incompressible fluid, equation (4.41) shows that the viscous stress at a point is

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.47}$$

which shows that σ depends only on the deformation rate of a fluid element at a point, and not on the rotation rate $(\partial u_i/\partial x_j - \partial u_j/\partial x_i)$. We have built this property into the Newtonian constitutive equation, based on the fact that in a solid-body rotation (that is a flow in which the tangential velocity is proportional to the radius) the particles do not deform or "slide" past each other, and therefore they do not cause viscous stress.

However, consider the net viscous force per unit volume at a point, given by

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = -\mu(\nabla \times \boldsymbol{\omega})_i, \tag{4.48}$$

where we have used the relation

$$(\nabla \times \boldsymbol{\omega})_i = \varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{kmn}\frac{\partial u_n}{\partial x_m}\right)$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial^2 u_n}{\partial x_j \partial x_m} = \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$
$$= -\frac{\partial^2 u_i}{\partial x_j \partial x_j}.$$