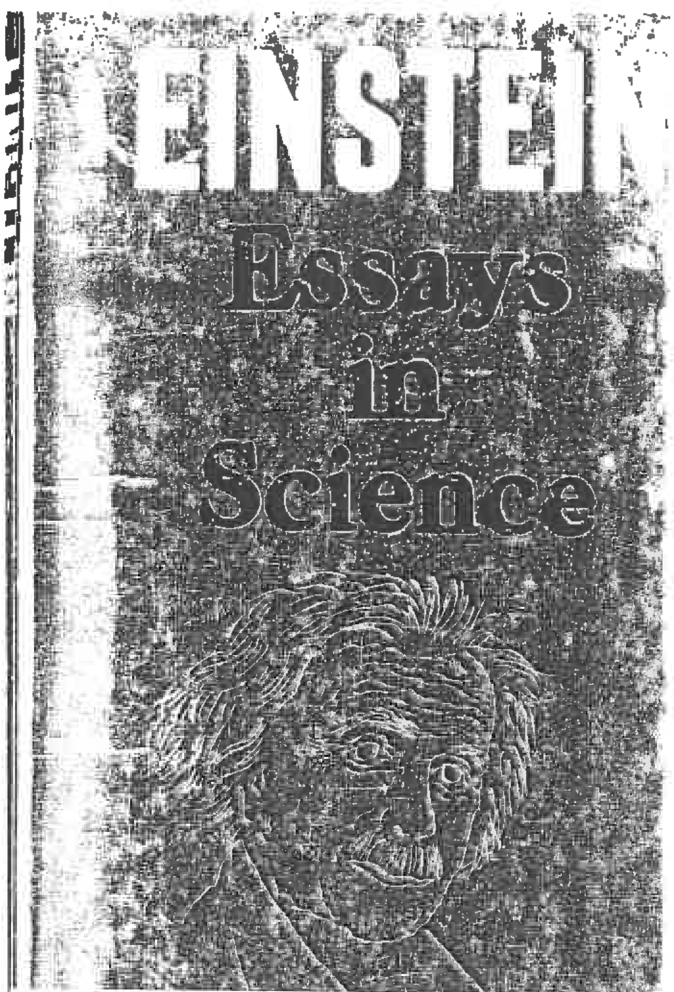
Lecture 8

Quick update from last time:

(1) Navier-Stokes Equations

(2) Bernoulli's eqn applied #2

plus energy Bernoulli addition



THE CAUSE OF THE FORMATION OF MEANDERS IN THE COURSES OF RIVERS AND OF THE SO-CALLED BEER'S LAW

It is common knowledge that streams tend to curve in serpentine shapes instead of following the line of the maximum declivity of the ground. It is also well known to geographers that the rivers of the northern hemisphere tend to erode chiefly on the right side. The rivers of the southern hemisphere behave in the opposite manner (Beer's law). Many attempts have been made to explain this phenomenon, and I am not sure whether anything I say in the following pages will be new to the expert; some of the relevant considerations are in any case known. Nevertheless, having found nobody who thoroughly understood the elementary principles involved, I think it is proper for me to give the following short qualitative exposition of them.

First of all, it is clear that the erosion must be stronger the greater the velocity of the current where it touches the bank in question, or the more steeply it falls to zero at any particular point of the confining wall. This is equally true under all circumstances, whether the erosion depends on mechanical or on

physico-chemical factors (decomposition of the ground). We must concentrate our attention on the circumstances which affect the steepness with which the velocity falls at the wall.

In both cases the asymmetry in relation to the fall in velocity in question is indirectly due to the occurrence of a circular motion to which we will next direct our attention. I begin with a little experiment which anybody can easily repeat.

Imagine a flat-bottomed cup full of tea. At the bottom there are some tea leaves, which stay there because they are rather heavier than the liquid they have displaced. If the liquid is made to rotate by a spoon, the leaves will soon collect in the center of the bottom of the cup. The explanation of this phenomenon is as follows:—The rotation of the liquid causes a centrifugal force to act on it. This in itself would give rise to no change in the flow of the liquid if the latter rotated like a solid body. But in the neighborhood of the walls of the cup the liquid is restrained by friction, so that the angular velocity with which it circulates is less there than in other places near the center. In particular, the angular velocity of circulation, and therefore the centrifugal force, will be smaller near the bottom than higher up. The result of this will be a circular movement of the liquid of the type illustrated in fig. 1. which goes on increasing until, under the influence of ground friction, it becomes stationary. The tea leaves are swept into the center by the circular movement and act as proof of its existence.

The same sort of thing happens with a curving stream (fig. 2). At every section of its course, where it is bent, a centrifugal force operates in the direction of the outside of the curve (from A to B). This force is less near the bottom, where the speed of the current is reduced by friction, than higher above the bottom. This causes a circular movement of the kind illustrated in the diagram. Even where there is

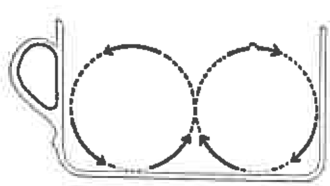


Fig. 1.

no bend in the river, a circular movement of the kind shown in fig. 2 will still take place, if only on a small scale and as a result of the earth's rotation. The latter produces a Coriolis-force, acting transversely to the direction of the current, whose right-hand horizontal component amounts to $2 v \Omega \sin \phi$ per unit of mass of the liquid, where v is the velocity of the current, Ω the speed of the earth's rotation, and ϕ the geographical latitude. As ground friction causes a diminution of this force towards the bottom, this force also gives rise to a circular movement of the type indicated in fig. 2.

After this preliminary discussion we come back to

Navier Stokes addition

Substituting $\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm}\delta_{ij}$

into

$$\tau_{ij} = p_n \delta_{ij} + \sigma_{ij}$$

with Stokes assumption that $\lambda + \frac{2}{3}\mu = 0$,

gives

$$(5\text{-}6) \qquad \tau_{ij} = -\left\{ p_n + \frac{2}{3}\mu \nabla\cdot u \right\}\delta_{ij} + 2\mu e_{ij}$$

$2\mu\partial_j\left(e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)\right)$

$= \mu\left(\partial^2 u_i + \partial_i \nabla\cdot u\right)$

$\mu\nabla^2\underline{u} + \mu\underline{\nabla}\,\underline{\nabla}\cdot\underline{u}$

Substituting into

Cauchy eqn

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial\tau_{ij}}{\partial x_j}$$

yields:

(5-8) Navier stokes equation (bounded - incompressible)

$$\boxed{\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p_n + \rho\underline{g} + \mu\nabla^2\underline{u}} + \frac{1}{3}\mu\underline{\nabla}\,\underline{\nabla}\cdot\underline{u}$$

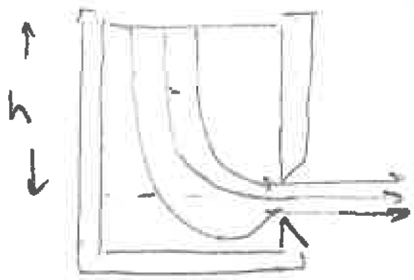
if flow is far from boundaries, viscous effect $\to 0$

and

Euler equation (5-9)

$$\boxed{\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \rho\underline{g}}$$

(2.) Orifice in a Tank



$$B = \text{constant along streamline}$$

$$= \frac{q^2}{2} + \frac{p}{\rho} + gz$$

$$= \frac{p_{atm}}{\rho} + gh \quad \text{at top}$$

$$= \frac{p_{atm}}{\rho} + \frac{u^2}{2} \quad \text{at jet}$$

because can use B, but or information

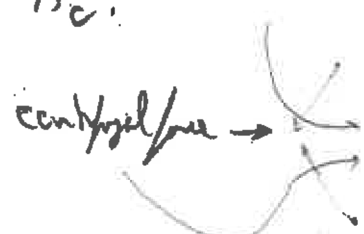
$$\therefore \quad u = \sqrt{2gh}$$

$$\dot{m} = \text{mass flux at} = \rho A_c u$$

$$\boxed{\dot{m} = \rho A_c \sqrt{2gh}}$$

note centrifugal force of converging streamlines

causes $A_c^{effect} \approx 0.62 \, A_c$.



→ [Also note Energy Bernoulli — total energy
$h + q^2/2 + gz$ const along streamline

Cauchy + constitutive → Navier Stokes, vorticity, Bernoulli, Energy " | Last time looked at the way forms of the conservation of momentum &

Lecture 8: Rotating frames

The final issue we need to consider (for much of the earth, but not to mantle) is the earth's rotation. Rotation gives rise to the coriolis force, which as anyone who has tried a rotating plate at a carnival knows, is truly a force to be reckoned with. Since our reference is the rotating surface of the earth, the coriolis force for oceanography and atmospheric dynamics is an important, in fact dominant, control.

The book does a good job of describing how the Navier-Stokes equation in a rotating frame changes from:

Navier Stokes

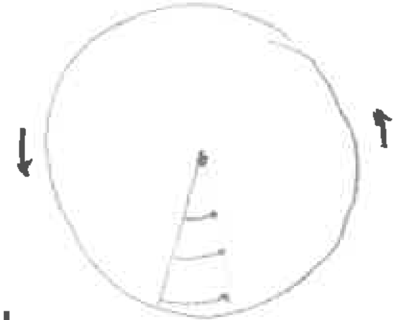
$$(8\text{-}1) \qquad \rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$$

where $\underline{u}$ is the velocity of a material element in the fixed frame, to

$$(8\text{-}2)\ (4.54\ \text{book}) \qquad \rho \frac{D\underline{u}}{Dt} + \underbrace{2\Omega \times \underline{u}}_{\text{coriolis accel}} - \underbrace{\Omega^2 \underline{R}}_{\text{centrifugal accel}} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$$

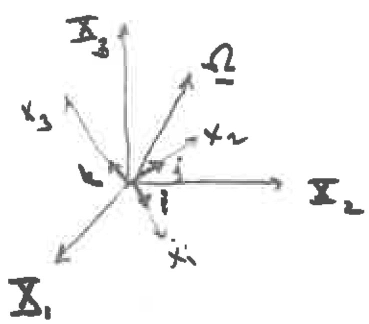
where $\underline{u}$ is the velocity of a material element in the rotating frame. The material frame physics is the same in both frames, but acceleration terms arise because $\underline{u}$ is now measured in the rotating frame.

Consider a rotating platform



- As you walk out you must accelerate to "walk the straight line" in the rotating frame
- your acceleration is perceived as a STRONG Force!
- We add this force to the Navier Stokes equations so it applies in the rotating frame

Consider a point $\underline{P}$ in fixed + rotating frames

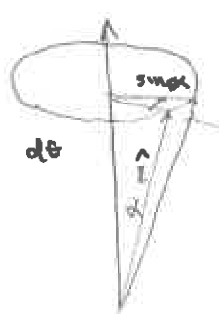


$$P_R = P_{1R}\,\hat{i} + P_{2R}\,\hat{j} + P_{3R}\,\hat{k}$$

$$\left(\frac{\partial P_R}{\partial t}\right)_F = \frac{d}{dt}\left(P_1\hat{i} + P_2\hat{j} + P_3\hat{k}\right)$$

$$= \hat{i}\frac{dP_1}{dt} + \hat{j}\frac{dP_2}{dt} + \hat{k}\frac{dP_3}{dt}$$

$$+ P_1\frac{d\hat{i}}{dt} + P_2\frac{d\hat{j}}{dt} + P_3\frac{d\hat{k}}{dt}$$



$\frac{d\hat{i}}{dt} =$

$\sin\alpha \frac{d\theta}{dt} =$

$\Omega \times i$

$$= \frac{d\underline{P}}{dt} + \underline{\Omega}\times\underline{P}$$

(2-3)
$$\boxed{\left(\frac{d\underline{P}}{dt}\right)_F = \left(\frac{d\underline{P}}{dt}\right)_R + \underline{\Omega}\times\underline{P}}$$

Apply to $\underline{P} = \underline{r}$

$$\underline{u}_F = \underline{u}_R + \underline{\Omega} \times \underline{r}$$

Apply to u_F

$$\left(\frac{d\underline{u}_F}{dt}\right)_F = \left(\frac{d\underline{u}_F}{dt}\right)_R + \underline{\Omega} \times \underline{u}_F$$

$$= \frac{d}{dt}\left(u_R + \underline{\Omega} \times \underline{r}\right)_R + \Omega \times \left(\underline{u}_R + \underline{\Omega} \times \underline{r}\right)$$

$$= \left(\frac{du_R}{dt}\right)_R + 2\underline{\Omega} \times \underset{\overset{\shortparallel}{\frac{dr_R}{dt}}}{\underline{u}_R} + \Omega \times (\Omega \times R)$$

$\perp$ axis rotation

$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$

$$\to -(\Omega \cdot \Omega)\underline{R} + (\cancel{\underline{\Omega} \cdot \underline{R}})\Omega$$

0

$$\left(\frac{du_F}{dt}\right)_F = \left(\frac{du_R}{dt}\right)_R + 2\Omega \times u_R - \Omega^2 R$$

$$a_F = a_R + 2\Omega \times u_R - \Omega^2 R$$

$\nu = \mu/\rho$

Thus the Navier Stokes equation in the Fixed Frame

$$\left(\frac{Du}{Dt}\right)_F = -\frac{1}{\rho}\nabla p + \underline{g} + \nu \nabla^2 \underline{u}$$

becomes in rotating frame

in rotational frame

(2.4)

$$\boxed{\left(\frac{Du}{Dt}\right)_R = -\frac{1}{\rho}\nabla p + \overbrace{\left(\underline{g} + \Omega^2 \underline{R}\right)}^{\text{effective gravity force}} + \nu \nabla^2 u - 2\Omega \times u}$$

Coriolis acceleration is just that required for a projectile to follow the path we all know it should.

Consider standing on the pole of rotation and firing south.

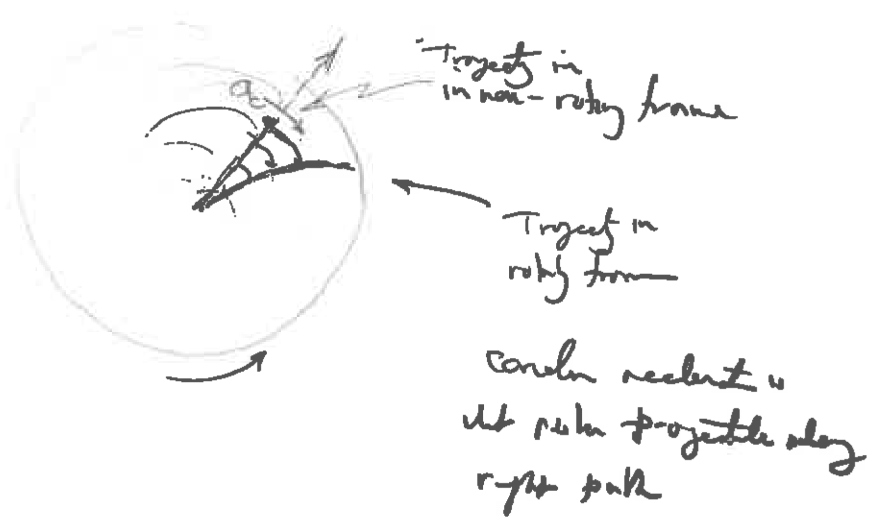
The projectile velocity u_r does not change. $\therefore x_s = ut$.

As projectile moves south rotation of earth $\uparrow$. $a_c = 2\Omega u$,

so $v_\theta = a_c t$, $x_\theta = \int a_c t \, dt = \frac{1}{2} a_c t^2 = \Omega u t^2$

$\therefore \alpha = \dfrac{\Delta x_\theta}{\Delta x_s} = \Omega t$ = just amount earth turned in t.

$\therefore$ earth turns under projectile. "Acceleration" arises from fact $u_\theta \uparrow$ with distance south. Acceleration defines the "fictitious" force need to "push" the projectile along the proper path.

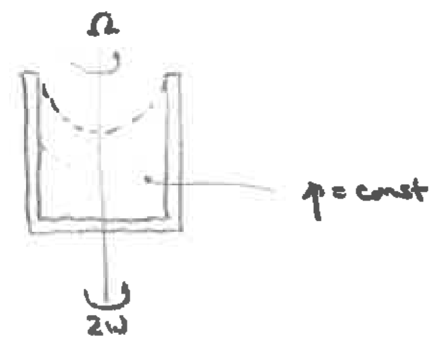


Vorticity

Incompressible Navier Stokes $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \underline{g} + \nu\nabla^2\underline{u} - 2\underline{\Omega}\times\underline{u}$$

Euler $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + g - 2\underline{\Omega}\times\underline{u}$$

Solid body rotation



From basic physics

$$F = \frac{\rho u_\theta^2}{r}$$

$$\frac{\partial p}{\partial r} = -\rho g \frac{\partial z}{\partial r}$$

$$\frac{\cancel{\rho} u_\theta^2}{r} - \cancel{\rho} g \frac{\partial z}{\partial r} = 0 \qquad \text{force balance}$$

substitute $u_\theta = \frac{1}{2}\omega r$

$$\int_{r_1}^{r_2} \frac{1}{4}\omega^2 r\,dr = \int_{z_1}^{z_2} g\,dz$$

$$\frac{1}{8}\frac{\omega^2}{g}\left(r_2^2 - r_1^2\right) = (z_2 - z_1)$$

Now from Euler equation

$$\frac{D\underline{u}}{Dt} = \frac{\partial u}{\partial t} + \boxed{\underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + g}$$

In polar coordinates (p 738) with $u_r = 0$, $u_\theta = \frac{1}{2}\omega_0 r$

$$\cancel{\frac{\partial u_r}{\partial t}}^{0} + u\cdot\nabla \cancel{u_r}^{0} \boxed{-\frac{u_\theta^2}{r} = \frac{-1}{\rho}\frac{\partial p}{\partial r}} + \nu\left(\cancel{\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}}^{0}\right)$$

$$\cancel{\frac{\partial u_\theta}{dt}}^{0} + \underbrace{u\cdot\nabla \cancel{u_\theta}^{0}}_{u_r\cdot\frac{1}{2}\omega_0 r} + \cancel{\frac{u_r u_\theta}{r}}^{0} = \frac{1}{\rho r}\cancel{\frac{\partial p}{\partial\theta}}^{0} + \nu\left(\cancel{\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}}^{0}\right)$$

$$\cancel{\frac{\partial u_z}{\partial t}}^{0} + (u\cdot\nabla)\cancel{u_z}^{0} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\cancel{\nabla^2 u_z} + g}$$

∂u_z

From p 738 $\sigma_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] = 0$

and of solid body rotation where $u_\theta = \omega_0 r$, $u_r = 0$, $\sigma_{r\theta} = 0$

Irrotational Vortex

$$u_\theta = \frac{\Gamma}{2\pi r} \qquad u_\theta = \frac{c}{r}, \quad u_r = 0$$

$$\omega = \underline{\nabla} \times \underline{u}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\mu (\nabla \times \omega)_i \qquad = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{rc}{r}\right)\right) = 0$$

since $\omega = 0$ there is no viscous force. ∴ again inviscid Euler equation holds

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

So

$$dp = \frac{\rho \Gamma^2}{4\pi^2 r^3} dr - \rho g \, dz$$

Swirls $\frac{1}{r}$ flow:

$$p_1 - p_2 = -\frac{\rho}{2}\left(u_{\theta 2}^2 - u_{\theta 1}^2\right) - \rho g (z_2 - z_1)$$

$$\frac{p_1}{\rho} + \frac{u_{\theta 1}^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{u_{\theta 2}^2}{2} + g z_2$$

Bernoulli equation valid between any two points (irrotational, not just along streamline)

$$\boxed{z_2 - z_1 = \frac{\Gamma}{8\pi^2 g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)}$$

Toilet drain

Kevin: Circulation Theorem

$$\frac{D\Gamma}{Dt} = 0$$

Circulation is conserved in a material element!

Circulation around a material contour does not change

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C u_i\, dx_i$$

$$= \int_C \frac{Du_i}{Dt}\, dx_i + \int_C u_i \frac{D}{Dt} dx_i$$

momentum equation

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho}\partial_j \sigma_{ij}$$

$$\int_C \frac{Du_i}{Dt}\, dx_i = -\int_C \frac{1}{\rho}\frac{\partial p}{\partial x_i}\, dx_i + \int g_i dx_i + \int \frac{1}{\rho}\partial_j \sigma_{ij}\, dx_j$$

$$\frac{D\Gamma}{Dt} = \int_C \underline{g}\cdot d\underline{x} - \int_C \frac{dp}{\rho} + \int_C \frac{1}{\rho}(\underline{\nabla}\cdot\underline{\underline{\sigma}})\cdot d\underline{x} + \int_C u_i \frac{D}{Dt} dx_i$$

$g = -\nabla\Pi$

$\Pi_A - \Pi_B$

if $A = B \Rightarrow 0$

barotropic

$\rho^{-1} = \frac{dP}{dp}$

$P_B - P_A \Rightarrow 0$

σ of viscous stresses can be neglected

$d\underline{u} = \frac{D}{Dt} d\underline{x}$

$\int_C d\left(\frac{1}{2}u_i\right)^2$

since u_i = same at start + end $\Rightarrow 0$

Create or destroy vortices:

Non-conservative body forces

Non-barotropic pressure-ρ relation

Viscous stresses

Implications

(1) Circulation is preserved

(2) Viscous effects cause diffusion of vorticity in + out of circuit

(3) Conservative body forces act through center of mass + do not produce vorticity

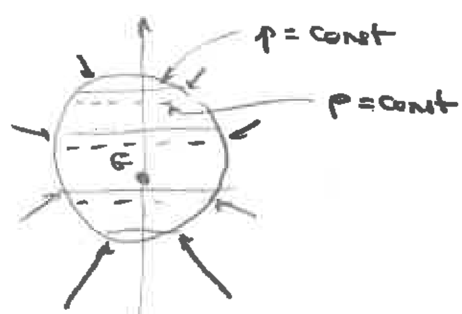
(4) Barotropic flow - inviscid

lines of const p parallel to lines of constant ρ so that ∇p acts at c of m

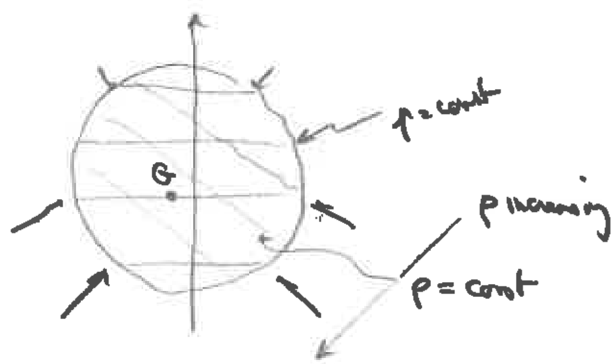
or

ρ(p) only (perfect gas $p/\rho^\gamma = \text{const}$)

not barotropic = baroclinic



G = center of mass



Pressure causes vorticity

Geophysical flows which are dominated by baroclinicity are full of vorticity

Heating - $\nabla_H \rho = 0$ necessary condition for static fluid

$\nabla_H \rho \neq 0$ generates vorticity

Helmholtz Theorems

(1) Vortex lines move with the fluid

(2) Strength of vortex tube (its circulation) = const along length

(3) Vortex tube cannot start or end
(end at bdry or be closed loop – vortex ring)

(4) Strength of a vortex tube remains constant in time

vortex tube enclosing material particles

S on boundary – circulation around edge of S = 0

as tube moves S will move but if stay with fluid $\Gamma_S = 0$

∴ no vorticity out of tube surface

∴ vorticity conserved in tube!

Vorticity equation

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}$$

$$= (\omega + 2\Omega)\cdot\nabla u + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\omega$$

stretching + tilting | baroclinic sourcing vorticity | diffusion

$$(\underline{\omega}\cdot\underline{\nabla})\underline{u} = \left[\underline{\omega}\cdot\left(i_s\frac{\partial}{\partial s} + i_n\frac{\partial}{\partial n} + i_m\frac{\partial}{\partial m}\right)\right]\underline{u} = \omega\frac{\partial u}{\partial s}$$

along a vortex line s

$\frac{\partial u}{\partial s}$ is stretching along vortex line

$\frac{\partial}{\partial n}, \frac{\partial}{\partial m}$ represent turning or tilting of vortex lines about m and n axes

$$\frac{D\underline{\omega}}{Dt} = \omega\frac{\partial u}{\partial s}$$

Stretching plays crucial role in dynamics of turbulent and geophysical flows →

$$\frac{D\omega_s}{Dt} = \omega\frac{\partial u_s}{\partial s}$$

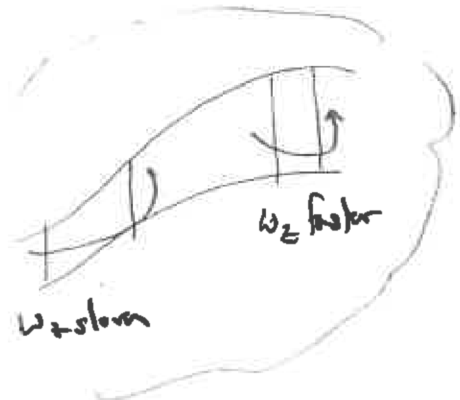
Vorticity changes in stretched vortex lines. stretching ↑ angular speed!

$$\frac{D\omega_n}{Dt} = \omega\frac{\partial u_n}{\partial s}$$

$$\frac{D\omega_m}{Dt} = \omega\frac{\partial u_m}{\partial s}$$

tilting shifts vorticity into other components

$$2(\underline{\Omega}\cdot\underline{\nabla})\underline{u}$$



if we orient z axis with $\underline{\Omega}$

$$\frac{D\underline{\omega}}{Dt} = 2\Omega\frac{\partial \underline{u}}{\partial z}$$

$$\frac{D\omega_z}{Dt} = 2\Omega\frac{\partial w}{\partial z}$$ Stretching in z increases ω_z

$$\frac{D\omega_x}{Dt} = 2\Omega\frac{\partial u}{\partial z}$$

$$\frac{D\omega_y}{Dt} = 2\Omega\frac{\partial v}{\partial z}$$

Tilting of vertical line changes relative vorticity in x and y directions

Do same vertical fluid lines contain "planetary vorticity" just turning of vertical fluid lines is enough — vortex lines do not need to tilt

Kelvin's circulation Theorem in rotating frame

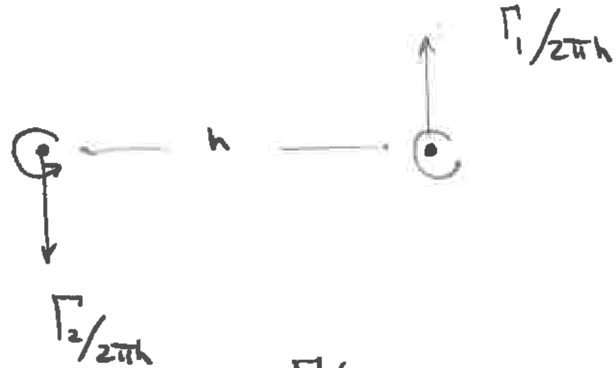
$$\frac{D\Gamma_a}{Dt} = 0$$

$$\Gamma_a = \int_A (\omega + 2\Omega)\cdot d\underline{A} = \Gamma + 2\underbrace{\int_A \Omega\cdot dA}_{\text{planetary vorticity}}$$

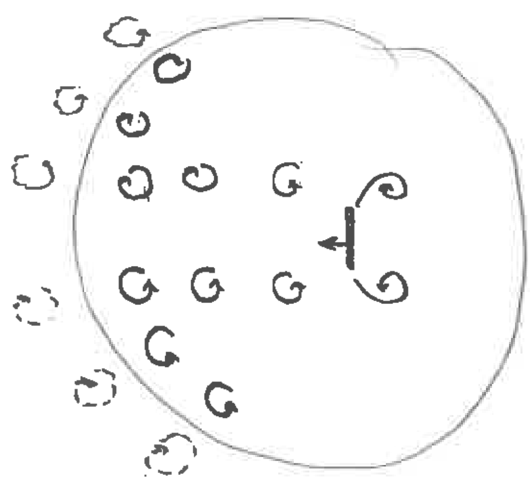
Interaction of Vortices —

vortices induce circulation at a distance $\left(\frac{1}{r}\right)$ and $\therefore$ interact

same sign rotate around each other



opposite sign translate



paddle generates vortices

Smoke Ring + vortex sheet

2 rings equal size + same rotation

- front slows - sucky of behind + ↑ in radius
- back speeds " of in front + ↓ in radius
- process rpt - yoyo

Vortex sheet - discontinuity of velocity - aircraft wing

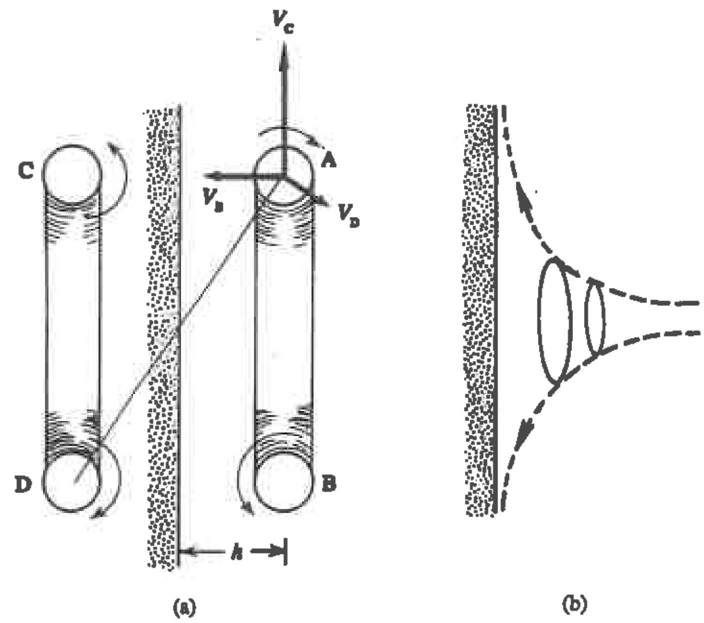


Figure 5.15 (a) Torus or doughnut-shaped vortex ring near a wall and its image. A section through the middle of the ring is shown. (b) Trajectory of vortex ring, showing that it widens while its translational velocity toward the wall decreases.

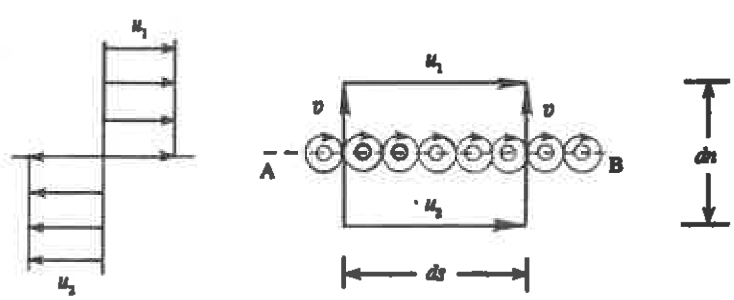


Figure 5.16 Vortex sheet.

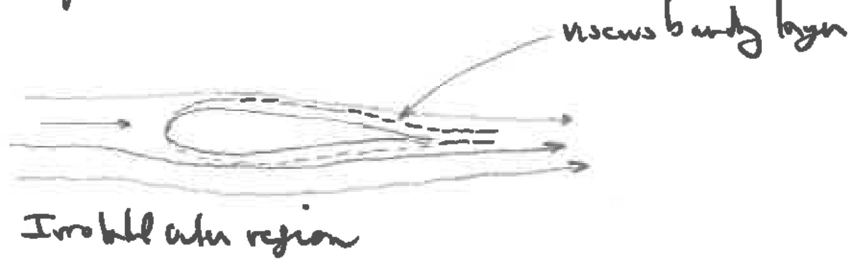
Chapter 6 - Irrotational flow

Irrotational flow stays irrotational
Rotational " " rotational

∴ Irrotational flow will only become rotational at a bdry

÷ 2 regions - near bdry — "Inner region"
- far from bdry — "outer region"

Eddie cayhede!



can use complex variables and law of superposition to compute ideal streamlines + vorticity lines

Add vortex circulation to flow around a circular body

streamline ↓ velocity potential ↓

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

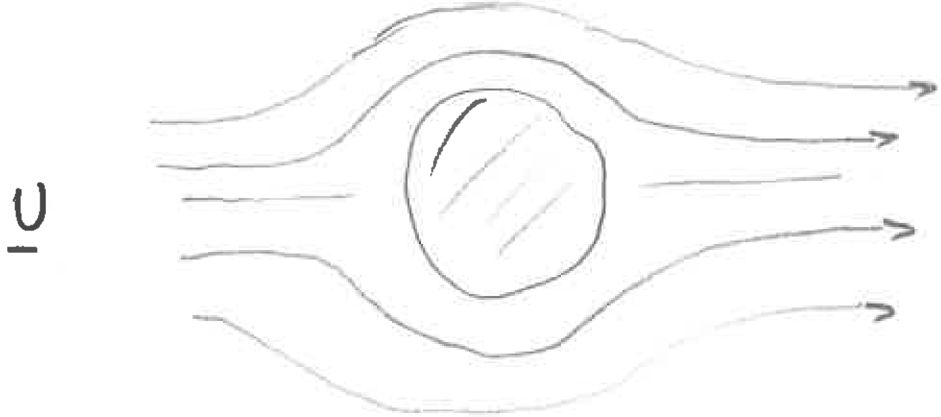
$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \text{Cauchy - Riemann condition}$$

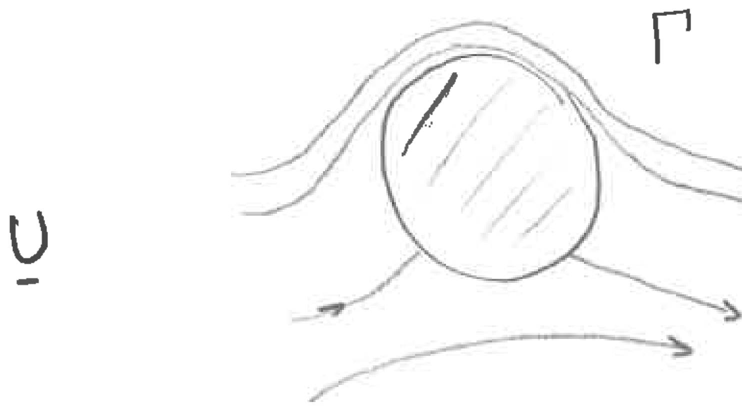
flow around sphere — vortex

$$w = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi} \ln z/a$$

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi} \ln r/a$$

$$z = x + iy = re^{i\theta}$$





With fluid vorticity added

Apply Bernoulli's principle to circuit around sphere

Kutta - Zhukhovsky lift Theorem

$$\boxed{L = \rho U \Gamma}$$

Lift

Lift is proportional to the circulation !

Viscosity generates Γ but magnitude of Γ indep of visc + depends only on U, shape + "attitude" of the foil

Magnus effect ↓

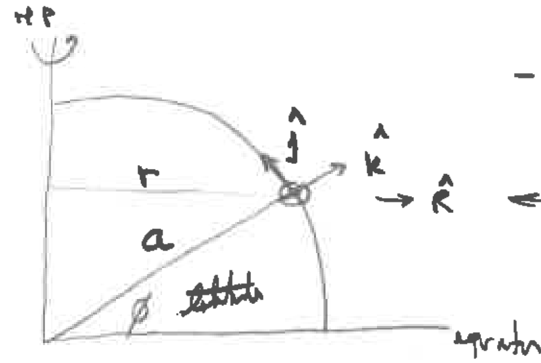
for circular body – must rotate to get lift.

golf ball dimples → delay of separation of eddies also impt

March 8, 2016

Mark Wysocki

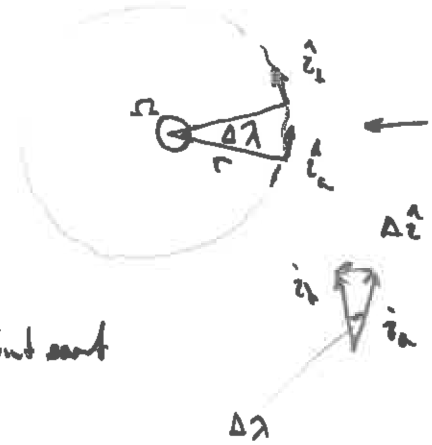
Side View



- Metrology convention

$\rightarrow \hat{R}$ ← unit vector in r direction

Top view



← longitude

$\Delta\hat{i} = \hat{i}\,\Delta\lambda$

$$u \frac{D\hat{i}}{Dt}$$

$\hat{i}$ = const - but chg w/ time unit vector

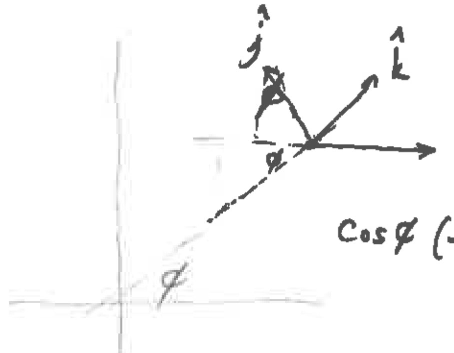
$$u\left(\cancel{\frac{\partial \hat{i}}{\partial t}} + u \cdot \frac{\partial \hat{i}}{\partial x} + \cancel{v \frac{\partial \hat{i}}{\partial y}} + \cancel{w \frac{\partial \hat{i}}{\partial z}}\right)$$

$\hat{i}$ point east

$$u \frac{\Delta i}{\Delta x} = -r\,\Delta\lambda$$

$$= -u \frac{i\,\Delta\lambda}{r\,\Delta\lambda} = -\frac{u}{r}\hat{R}$$

$\sin\phi\ \hat{j} = -\hat{R}.$



$\cos\phi\,(-\hat{k}) = -\hat{R}$

$$\boxed{u \frac{Di}{Dt} = \frac{u^2}{a}\tan\phi\ \hat{j} - \frac{u^2}{a}\hat{k}}$$

$$\hat{R} = \left(-\sin\phi\ \hat{j} + \cos\phi\ \hat{k}\right)$$

$$\frac{\partial \hat{i}}{\partial x} = \frac{u}{a}\tan\phi\,\hat{j} - \frac{u}{a}\hat{k}$$

$$\frac{\partial \hat{i}}{\partial x} = \frac{1}{r}u\left(\sin\phi\ \hat{j} - \cos\phi\ \hat{k}\right) = \frac{\partial \hat{i}}{\partial x} = \frac{u}{a\cos\phi}\left(\sin\phi\ \hat{j} - \cos\phi\ \hat{k}\right)$$

$= a\cos\phi$

Typically on a Synoptic scale

4 8 hr — W coast to E coast

$U = 10 \, m \, s^{-1}$ typically

$a = 6.37 \times 10^6 \, m$

$\phi = 45°$

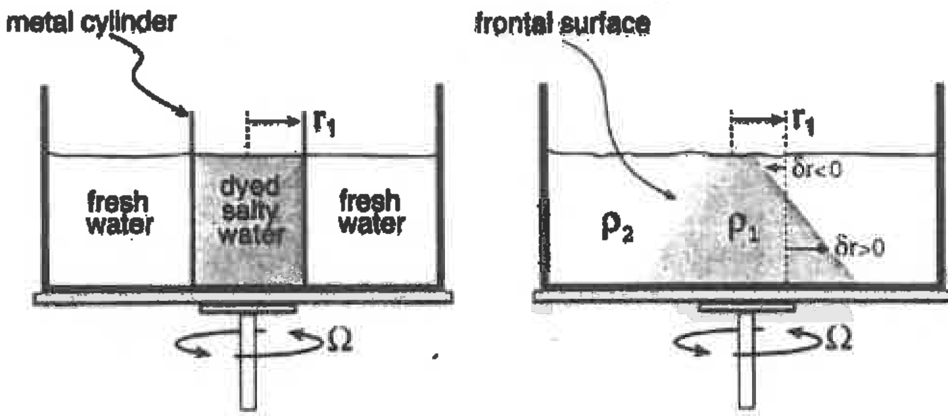
$$u \frac{D_i}{Dt} = \frac{u}{a} \tan\phi \, j - \frac{u}{a} \, k$$

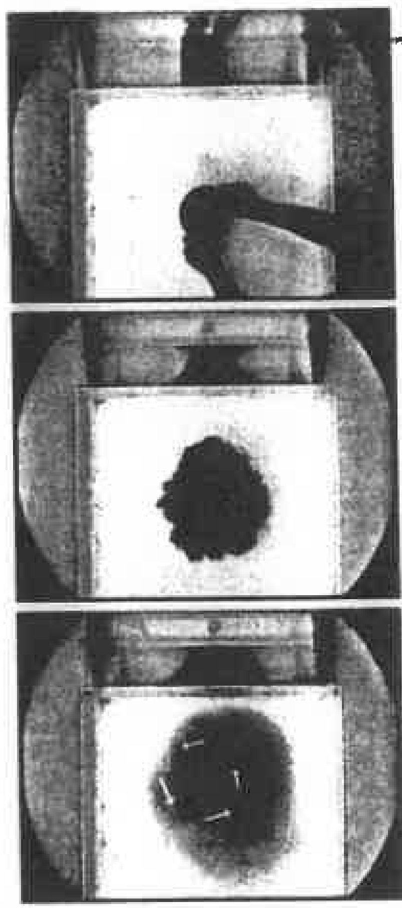
$$= 1.57 \times 10^{-6} \, m \, s^{-2} - 1.5 \times 10^{-6} \, m \, s^{-2}$$

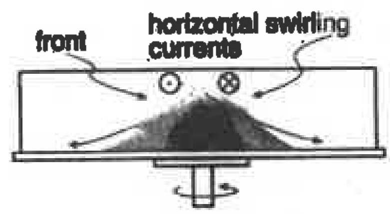
$\cong \phi$ in mid ~~latitudes~~
cancel !

GCMs must take into account.

Geostrophic flow — ∇p balanced by coriolis force

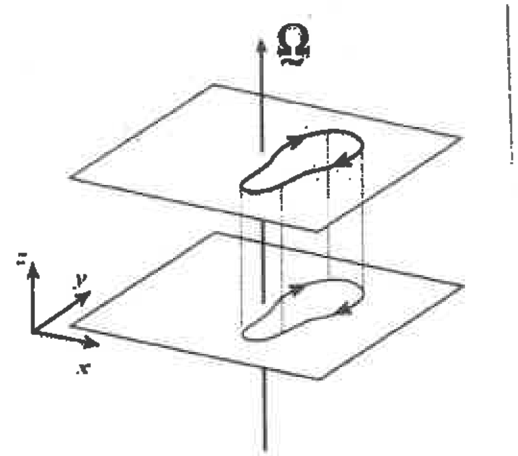


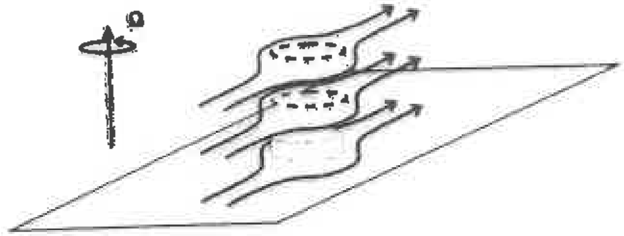


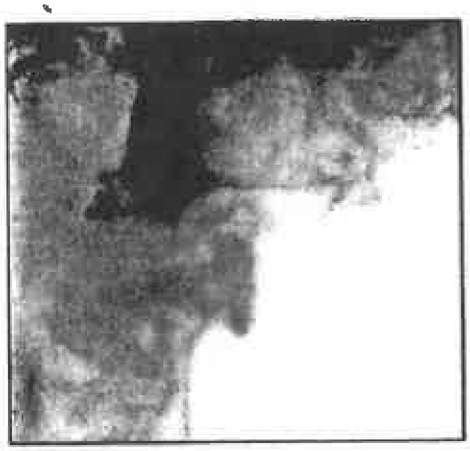


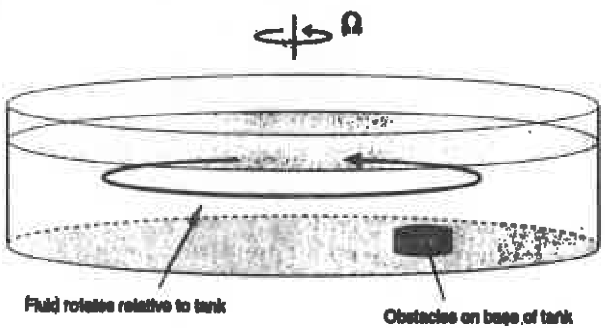
salt water sinking under gravity and rotation

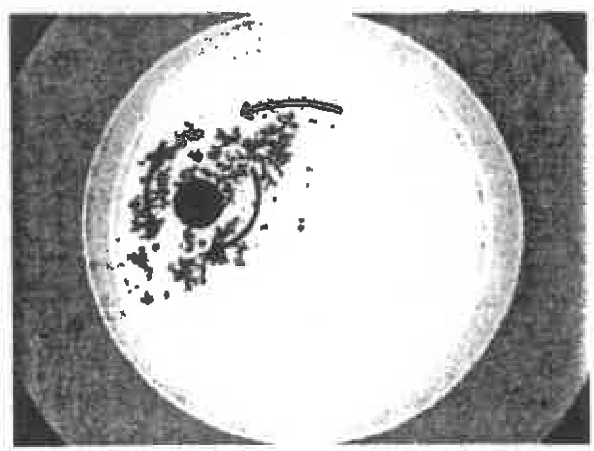
From Marshall + Plumb, 2008











From Marshall & Plumb, [illegible]