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# Polynomiality of primal-dual algorithms for semidefinite linear complementarity problems based on the Kojima-Shindoh-Hara family of directions

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**Abstract.** Kojima, Shindoh and Hara proposed a family of search directions for the semidefinite linear complementarity problem (SDLCP) and established polynomial convergence of a feasible short-step path-following algorithm based on a particular direction of their family. The question of whether polynomiality could be established for any direction of their family thus remained an open problem. This paper answers this question in the affirmative by establishing the polynomiality of primal-dual interior-point algorithms for SDLCP based on any direction of the Kojima, Shindoh and Hara family of search directions. We show that the polynomial iteration-complexity bounds of two well-known algorithms for linear programming, namely the short-step path-following algorithm of Kojima et al. and Monteiro and Adler, and the predictor-corrector algorithm of Mizuno et al., carry over to the context of SDLCP.

**Key words.** semidefinite programming – interior-point methods – polynomial complexity – path-following methods – primal-dual methods

## 1. Introduction

Several authors have discussed generalizations of interior-point algorithms for linear programming (LP) to the context of semidefinite programming (SDP) and the more general semidefinite linear complementarity problem (SDLCP). The landmark work in this direction is due to Nesterov and Nemirovskii [1,2] where a general approach for using interior-point methods for solving convex programs is proposed based on the notion of self-concordant functions. (See their book [3] for a comprehensive treatment of this subject.) They show that the problem of minimizing a linear function over a convex set can be solved in “polynomial time” as long as a self-concordant barrier function for the convex set is known. In particular, Nesterov and Nemirovskii show that linear programs, convex quadratic programs with convex quadratic constraints, and semidefinite programs all have explicit and easily computable self-concordant barrier functions, and hence can be solved in “polynomial time”. On the other hand, Alizadeh [4] extends Ye’s projective potential reduction algorithm [5] for LP to SDP and argues that many known interior point algorithms for LP can also be transformed into algorithms for SDP in a mechanical way. Since then many authors have proposed interior-point algorithms for solving the SDP problem and SDLCP, including Alizadeh, Haeberly and

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Overton [6], Freund [7], Helmberg, Rendl, Vanderbei and Wolkowicz [8], Jarre [9], Kojima, Shida and Shindoh [10], Kojima, Shindoh and Hara [11], Lin and Saigal [12], Luo, Sturm and Zhang [13], Monteiro [14,15], Monteiro and Zhang [16], Monteiro and Tsuchiya [17], Monteiro and Zanjácomo [18], Nesterov and Nemirovskii [19], Nesterov and Todd [20,21], Potra and Sheng [22], Sturm and Zhang [23], Tseng [24], Vandenberghe and Boyd [25], and Zhang [26]. Most of these more recent works are concentrated on primal-dual methods.

The first algorithms for SDP and SDLCP that are extensions of primal-dual algorithms for LP, such as the long-step path-following algorithm of Kojima, Mizuno and Yoshise [27], the short-step path-following algorithm of Kojima, Mizuno and Yoshise [28] and Monteiro and Adler [29,30], and the predictor-corrector algorithm of Mizuno, Todd and Ye [31], use one of the following three search directions: i) the Alizadeh, Haeberly and Overton (AHO) direction proposed in [6], ii) the HRVW/KSH/M direction independently proposed by Kojima, Shindoh and Hara [11] and Helmberg, Rendl, Vanderbei and Wolkowicz [8], and later rediscovered by Monteiro [14] via a formulation based on a scaling and symmetrization of the Newton equation, and iii) the Nesterov and Todd (NT) direction introduced in [20,21].

Several families of search directions have been proposed in the literature in an attempt to study primal-dual algorithms for SDP in a unified manner. The first family, proposed by Kojima, Shindoh and Hara [11], is known to contain the HRVW/KSH/M and NT directions but not the AHO direction. The second family, namely the Monteiro and Zhang (MZ) family, formally introduced by Zhang [26] to generalize a symmetric formulation of the HRVW/KSH/M direction proposed by Monteiro [14], contains all three search directions above. Proofs that the NT direction is a member of both the KSH family and the MZ family can be found in Kojima, Shida and Shindoh [32] and Todd, Toh and Tütüncü [33], respectively. The third family, namely the Monteiro and Tsuchiya (MT) family introduced in [17] shortly after the release of the first version of this paper, is based on a different representation of the central path that is directly related to the centrality measures used in standard path following algorithms. This family also contains the HRVW/KSH/M and NT directions (but not the AHO direction). Finally, we mention that Tseng [24] also considers a family of search directions parametrized by a single scalar parameter which contains the NT and HRVW/KSH/M directions.

Unified convergence analyses for the MZ family have been given by Monteiro and Zhang [16] and Monteiro [15]. In the paper [16], iteration-complexity bounds are derived for long-step primal-dual path-following methods based on a subclass of the MZ family of search directions, which contains the HRVW/KSH/M and NT directions but not the AHO direction. In particular, it is shown that the corresponding algorithms based on the HRVW/KSH/M and NT directions perform $\mathcal{O}(n^{3/2}\log\varepsilon^{-1})$ and $\mathcal{O}(n\log\varepsilon^{-1})$ iterations, respectively, to reduce the duality gap by a factor of at least $\varepsilon$. (The $\mathcal{O}(n^{3/2}\log\varepsilon^{-1})$ iteration-complexity bound for the HRVW/KSH/M direction was in fact obtained earlier by Monteiro [14].) More recently, Monteiro [15] proves the polynomiality of short-step path following algorithms and Mizuno-Todd-Ye predictor-corrector type algorithms based on any member of the MZ family, thus obtaining as a by-product the important result that Frobenius-norm type algorithms based on the AHO direction are polynomial.

For the MT family, Monteiro and Tsuchiya [17] establish $\mathcal{O}(\sqrt{n}\log\varepsilon^{-1})$ and $\mathcal{O}(n\log\varepsilon^{-1})$ iteration-complexity bounds for the short-step and semilong-step path following algorithms, respectively [17]. They also consider a subclass of the MT family which contains the HRVW/KSH/M and NT directions, and establish an $\mathcal{O}(n^{3/2} \log \varepsilon^{-1})$ iteration complexity bound for any long-step path following algorithm based on this subclass. Monteiro and Zanjácomo [18] report promising computational results for algorithms based on some directions of the MT family.

Unified analysis for the KSH family of search directions are provided in Kojima, Shindoh and Hara [11]. This paper deals with primal-dual path-following algorithms for the semidefinite linear complementarity problem based on the KSH family of search directions and establishes the polynomiality of: 1) a feasible short-step path-following method based on a *special* member of their family, namely the HRVW/KSH/M direction and; 2) a (feasible and infeasible) potential reduction algorithm based on *any* search direction of their family. The question of whether polynomiality of algorithm 1) can be established for any direction of the KSH family was thus left as an open problem. (N. B. The polinomiality results for the MZ family and the MT family do not provide answer to this question because these families have overlaps one another, but neither of them includes others.)

In this paper, we answer the above question in the affirmative. Using new techniques recently proposed by Monteiro [15], we prove the polynomial convergence of two feasible primal-dual algorithms based on a narrow (or Frobenius norm) neighborhood of the central path, namely: a short-step path-following method which is an extension of the LP method of Kojima, Mizuno and Yoshise [28] and Monteiro and Adler [29,30], and a predictor-corrector algorithm similar to the LP one of Mizuno, Todd and Ye [31].

This paper is organized as follows. In Section 2, we introduce the SDLCP problem and motivate the search directions used by the algorithms studied in this paper. In Section 3, we state and prove the technical results used in the polynomial convergence analysis of the algorithms of Section 4. In Section 4, we establish the polynomiality of two primal-dual feasible algorithms: the short-step path-following algorithm in Subsection 4.1 and the predictor-corrector algorithm in Subsection 4.2. We give some concluding remarks in Section 5.

### *1.1. Notation and terminology*

The following notation is used throughout the paper. The superscript $^T$ denotes transpose. $\Re^p$ denotes the $p$-dimensional Euclidean space. The set of all $p \times q$ matrices with real entries is denoted by $\Re^{p\times q}$. The set of all symmetric $p \times p$ matrices is denoted by $\mathcal{S}^p$. For $Q \in \mathcal{S}^p$, $Q \succeq 0$ means $Q$ is positive semidefinite and $Q \succ 0$ means $Q$ is positive definite. The trace of a matrix $Q \in \Re^{p\times p}$ is denoted by $\mathrm{Tr}\ Q \equiv \sum_{i=1}^n Q_{ii}$. For a matrix $Q \in \Re^{p\times p}$ with all real eigenvalues, we denote its eigenvalues by $\lambda_i[Q]$, $i = 1, \dots, p$, and its smallest eigenvalue by $\lambda_{\min}[Q]$. Given $P$ and $Q$ in $\Re^{p\times q}$, the inner product between them in the vector space $\Re^{p\times q}$ is defined as $P \bullet Q \equiv \mathrm{Tr}\ P^T Q$. The Euclidean norm and its associated operator norm are both denoted by $\|\cdot\|$; hence, $\|Q\| \equiv \max_{\|u\|=1} \|Qu\|$ for any $Q \in \Re^{p\times p}$. The Frobenius norm of $Q \in \Re^{p\times p}$ is $\|Q\|_F \equiv (Q \bullet Q)^{1/2}$. We frequently use the inequalities $\|Q\| \le \|Q\|_F$ and $\|QR\|_F \le \|Q\|\|R\|_F$, for $Q, R \in \Re^{p\times p}$. $\mathcal{S}^p_+$ and $\mathcal{S}^p_{++}$ denote the set of all matrices in $\mathcal{S}^p$ which are positive semidefinite and positive definite, respectively. $\mathcal{S}^p_\perp$ denotes the set of all skew-symmetric matrices in $\Re^{p\times p}$. Since $\mathcal{S}^p + \mathcal{S}^p_\perp = \Re^{p\times p}$ and $U \bullet V = 0$ for every $U \in \mathcal{S}^p$ and $V \in \mathcal{S}^p_\perp$, it follows that $\mathcal{S}^p_\perp$ is the orthogonal complement of $\mathcal{S}^p$ with respect to the inner product $\bullet$.

## 2. Description of the problem and preliminary discussion

In this section, we introduce the semidefinite linear complementarity problem and the assumptions made in our presentation. We also describe the family of search directions introduced by Kojima, Shindoh and Hara [11] and give a short proof for the existence and uniqueness of these directions.

Let $\mathcal{L}$ be an affine subspace of $\mathcal{S}^n \times \mathcal{S}^n$ whose dimension is $n(n+1)/2$. Let

$$\mathcal{L}_+ \equiv \mathcal{L} \cap (\mathcal{S}^n_+ \times \mathcal{S}^n_+),$$
$$\mathcal{L}_{++} \equiv \mathcal{L} \cap (\mathcal{S}^n_{++} \times \mathcal{S}^n_{++}).$$

In this paper, we deal with the semidefinite linear complementarity problem (SDLCP) of finding a pair $(X, S)$ such that

$$(X, S) \in \mathcal{L}_+, \quad X \bullet S = 0. \tag{1}$$

Throughout our presentation, we assume that

**[A1]** $\mathcal{L}$ is monotone, that is $(X_1 - X_2) \bullet (S_1 - S_2) \geq 0$ for any $(X_1, S_1) \in \mathcal{L}$ and $(X_2, S_2) \in \mathcal{L}$.

**[A2]** $\mathcal{L}_{++}$ is nonempty.

This problem includes SDP which has numerous applications in systems and control theory and combinatorial optimization. Given $C \in \mathcal{S}^n$ and $(A_i, b_i) \in \mathcal{S}^n \times \Re$ for $i = 1, \ldots, m$, a primal-dual pair of SDP problems is defined as

$$(P) \quad \min\{C \bullet X : A_i \bullet X = b_i,\ i = 1, \ldots, m,\ X \succeq 0\},$$
$$(D) \quad \max\{b^T y : \sum_{i=1}^m y_i A_i + S = C,\ S \succeq 0\},$$

where $b \equiv (b_1, \ldots, b_m)^T$. Under the assumption that problems $(P)$ and $(D)$ have interior feasible solutions, that is feasible solutions $X$ and $(S, y)$ satisfying $X \succ 0$ and $S \succ 0$, it is known that $(X, S)$ is a solution of (1) with

$$\mathcal{L} = \{(X, S) \in \mathcal{S}^n \times \mathcal{S}^n : A_i \bullet X = b_i,\ i = 1, \ldots, m,$$
$$\sum_{i=1}^m y_i A_i + S = C \text{ for some } y \in \Re^m\},$$

if and only if $(X, S, y)$ is a solution of $(P)$ and $(D)$ for some $y \in \Re^m$. In this case, it is easy to see that $\mathcal{L}$ is a monotone affine space satisfying $(X_1 - X_2) \bullet (S_1 - S_2) = 0$ for any $(X_1, S_1) \in \mathcal{L}$ and $(X_2, S_2) \in \mathcal{L}$.

Under assumptions [A1] and [A2], it is known that problem (1) has at least one solution. Since for $(X, S) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+$, we have $X \bullet S = 0$ if and only if $XS = 0$, problem (1) is equivalent to find a pair $(X, S)$ such that

$$(X, S) \in \mathcal{L}_+, \quad XS = 0.$$

It has been shown by Kojima, Shindoh and Hara [11] that the perturbed system

$$(X, S) \in \mathcal{L}_+, \quad XS = \mu I, \tag{2}$$

has a unique solution in $\mathcal{L}_{++}$, denoted by $(X_\mu, S_\mu)$, for every $\mu > 0$, and that $\lim_{\mu\to 0}(X_\mu, S_\mu)$ exists and is a solution of (1). The set $\{(X_\mu, S_\mu) : \mu > 0\}$ is called the central path associated with (1) and plays a fundamental role in the development of interior point algorithms for solving SDP and SDLCP. Another equivalent formulation of (2) is

$$(X, S) \in \mathcal{L}_+, \quad X^{1/2}SX^{1/2} = \mu I \ \text{(or, } S^{1/2}XS^{1/2} = \mu I),$$

which motivates the following measure of closeness of $(X, S) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+$ to the point $(X_\mu, S_\mu)$ of the central trajectory:

$$d_\mu(X, S) \equiv \left\| X^{1/2}SX^{1/2} - \mu I \right\|_F = \left\| S^{1/2}XS^{1/2} - \mu I \right\|_F,$$

and the following (feasible) neighborhood of $(X_\mu, S_\mu)$:

$$\mathcal{N}_F(\mu, \gamma) = \left\{ (X, S) \in \mathcal{L}_+ : d_\mu(X, S) \le \gamma\mu \right\},$$

where $\gamma > 0$ is a given constant. Both algorithms described in Section 4 generate their iterates in the neighborhood of the central path defined by

$$\mathcal{N}_F(\gamma) \equiv \cup_{\mu>0}\mathcal{N}_F(\mu, \gamma).$$

Path-following algorithms for solving (1) are based on the idea of approximately tracing the central path. Application of Newton method for computing the solution of (2) with $\mu = \hat{\mu}$ leads to the Newton search direction $(\widehat{\Delta X}, \widehat{\Delta S})$ which solves the linear system

$$X\widehat{\Delta S} + \widehat{\Delta X}S = \hat{\mu}I - XS, \quad (X + \widehat{\Delta X}, S + \widehat{\Delta S}) \in \mathcal{L}. \tag{3}$$

Unfortunately, this system does not always have a solution. To overcome this difficulty, Kojima and Shindoh and Hara proposed the following modified Newton system of equations:

$$X(\Delta S + \widetilde{\Delta S}) + (\Delta X + \widetilde{\Delta X})S = \hat{\mu}I - XS, \tag{4a}$$
$$(X + \Delta X, S + \Delta S) \in \mathcal{L}, \quad (\widetilde{\Delta X}, \widetilde{\Delta S}) \in \mathcal{L}_\perp, \tag{4b}$$

where $\mathcal{L}_\perp$ is a linear subspace of $\Re^{n\times n} \times \Re^{n\times n}$ satisfying the following condition:

**[A3]** $\mathcal{L}_\perp \subseteq \mathcal{S}^n_\perp \times \mathcal{S}^n_\perp$, $\dim(\mathcal{L}_\perp) = n(n-1)/2$ and $\mathcal{L}_\perp$ is monotone, that is $U \bullet V \ge 0$ for every $(U, V) \in \mathcal{L}_\perp$.

It was shown in Corollary 4.3 of [11] that system (4) always has a unique solution. The symmetric component $(\Delta X, \Delta S)$ of this solution is then used as a search direction to generate the next point. In what follows we give another short proof of the existence and uniqueness of $(\Delta X, \widetilde{\Delta X}, \Delta S, \widetilde{\Delta S})$, which gives some intuition for the need to introduce the subspace $\mathcal{L}_\perp$.

**Lemma 1.** *Let* $(X, S) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ *and* $\mathcal{W}$ *be an* $n^2$ *dimensional affine subspace of* $\Re^{n\times n} \times \Re^{n\times n}$ *which is monotone, that is* $(U_1 - U_2) \bullet (V_1 - V_2) \geq 0$ *for every* $(U_1, V_1), (U_2, V_2) \in \mathcal{W}$. *Then, the system*

$$XV + US = H, \quad (U, V) \in \mathcal{W}, \tag{5}$$

*has a unique solution for any* $H \in \Re^{n\times n}$.

*Proof.* Consider the map $\Phi : \mathcal{W} \to \Re^{n\times n}$ defined by $\Phi(U, V) = XV + US$ for every $(U, V) \in \mathcal{W}$. $\Phi$ is an affine map between spaces of the same dimension since $\dim(\mathcal{W}) = n^2$ by assumption. Hence, it suffices to show that $\Phi$ is one-to-one. Indeed, assume that $\Phi(U_1, V_1) = \Phi(U_2, V_2)$ for some $(U_1, V_1), (U_2, V_2) \in \mathcal{W}$. Letting $\Delta U \equiv U_1 - U_2$ and $\Delta V \equiv V_1 - V_2$, and using the monotonicity of $\mathcal{W}$, we see that $\Delta U \bullet \Delta V \geq 0$ and $X\Delta V + \Delta U S = 0$. Multiplying the last relation on the left by $X^{-1/2}$ and on the right by $S^{-1/2}$, squaring both sides and using the fact that $\Delta U \bullet \Delta V \geq 0$, we obtain

$$0 = \left\| X^{1/2}\Delta V S^{-1/2} + X^{-1/2}\Delta U S^{1/2} \right\|_F^2 \geq \left\| X^{1/2}\Delta V S^{-1/2} \right\|_F^2 + \left\| X^{-1/2}\Delta U S^{1/2} \right\|_F^2.$$

Hence, $\Delta U = \Delta V = 0$, or equivalently $(U_1, V_1) = (U_2, V_2)$.

□

Lemma 1 provides the main reason for system (3) to not always have a solution, namely: the solution $(\widehat{\Delta X}, \widehat{\Delta S})$ is required to belong to the affine subspace $\mathcal{L} - (X, S)$, which only has dimension $n(n+1)/2 < n^2$. Adding the subspace $\mathcal{L}_\perp$ to $\mathcal{L}$ results in an affine subspace of dimension $n^2$ as required by Lemma 1. This fact is exploited in the proof of the following result which establishes the existence and uniqueness of the solution of (4).

**Theorem 1.** *System (4) has a unique solution.*

*Proof.* It is easy to see that $(\Delta X, \widetilde{\Delta X}, \Delta S, \widetilde{\Delta S})$ is a solution of (4) if and only if $(U, V) \equiv (\Delta X + \widetilde{\Delta X}, \Delta S + \widetilde{\Delta S})$ is a solution of (5) with $\mathcal{W} \equiv (\mathcal{L} - (X, Y)) + \mathcal{L}_\perp$ and $H \equiv \hat{\mu} I - XS$. Since $\mathcal{L}$ and $\mathcal{L}_\perp$ are monotone and orthogonal, $\dim(\mathcal{L}) = n(n+1)/2$ and $\dim(\mathcal{L}_\perp) = n(n-1)/2$, we easily see that $\mathcal{W}$ is a monotone affine subspace of $\Re^{n\times n} \times \Re^{n\times n}$ of dimension $n^2$. The result now follows from Lemma 1.

□

## 3. Technical results

In this section we provide some technical results which will be used to establish the polynomial convergence of the algorithms presented in Section 4.

We assume throughout this section that $(X, S) \in \mathcal{L}_{++}$ and that $(\Delta X, \widetilde{\Delta X}, \Delta S, \widetilde{\Delta S})$ is a solution of system (4) with $\hat{\mu} = \sigma\mu$ for some $\mu > 0$ and $\sigma \in [0, 1]$. Moreover, we define for every $\alpha \in \Re$,

$$X(\alpha) \equiv X + \alpha\Delta X, \quad S(\alpha) \equiv S + \alpha\Delta S, \tag{6}$$

$$\mu(\alpha) \equiv (1 - \alpha + \sigma\alpha)\mu. \tag{7}$$

**Lemma 2.** *For every $\alpha \in \Re$, we have*

$$X(\alpha)S(\alpha) - \mu(\alpha)I = (1-\alpha)\,(XS - \mu I) - \alpha\,\left(X\widetilde{\Delta S} + \widetilde{\Delta X}S\right) + \alpha^2 \Delta X \Delta S. \tag{8}$$

*Proof.* Follows immediately from (6), (7) and (4a) with $\hat{\mu} = \sigma\mu$.

□

For a nonsingular matrix $P \in \Re^{n\times n}$, consider the following operator $H_P : \Re^{n\times n} \to \mathcal{S}^n$ defined as

$$H_P(M) \equiv \frac{1}{2}\left[PMP^{-1} + (PMP^{-1})^T\right], \quad \forall M \in \Re^{n\times n}.$$

The operator $H_P$ has been recently used by Zhang [26] to characterize the central path of SDP problems.

**Lemma 3.** *For every $\theta \in \Re$ and $\alpha \in [0, 1]$, we have*

$$\begin{aligned}\left\| H_{X^{-1/2}}\left[X(\alpha)S(\alpha) - \mu(\alpha)I\right]\right\|_F \le (1-\alpha)\,\left\|X^{1/2}SX^{1/2} - \mu I\right\|_F + \alpha^2\,\delta_x\,\delta_s \\ + \alpha\,\tilde{\delta}_x\,\left\|X^{1/2}SX^{1/2} - \theta\mu I\right\|,\end{aligned} \tag{9}$$

*where*

$$\delta_x \equiv \left\|X^{-1/2}\Delta X X^{-1/2}\right\|_F, \quad \tilde{\delta}_x \equiv \left\|X^{-1/2}\widetilde{\Delta X} X^{-1/2}\right\|_F, \quad \delta_s \equiv \left\|X^{1/2}\Delta S X^{1/2}\right\|_F. \tag{10}$$

*Proof.* Using the fact that $\widetilde{\Delta X}, \widetilde{\Delta S} \in \mathcal{S}^n_\perp$, and hence that $H_I(\widetilde{\Delta X}) = H_I(\widetilde{\Delta S}) = 0$, we obtain

$$\begin{aligned}&H_{X^{-1/2}}\left(X\widetilde{\Delta S} + \widetilde{\Delta X}S\right)\\ &= H_{X^{-1/2}}\left(X\widetilde{\Delta S} + \theta\mu\widetilde{\Delta X}X^{-1}\right) + H_{X^{-1/2}}\left[\widetilde{\Delta X}\left(S - \theta\mu X^{-1}\right)\right]\\ &= X^{1/2}H_I(\widetilde{\Delta S})X^{1/2} + \theta\mu X^{-1/2}H_I(\widetilde{\Delta X})X^{-1/2} + H_{X^{-1/2}}\left[\widetilde{\Delta X}\left(S - \theta\mu X^{-1}\right)\right]\\ &= H_I\left[X^{-1/2}\widetilde{\Delta X}X^{-1/2}\left(X^{1/2}SX^{1/2} - \theta\mu I\right)\right].\end{aligned}$$

Using (8), the last relation, (10) and the inequality $\|H_I(M)\|_F \le \|M\|_F$ for $M \in \Re^{n\times n}$, we obtain for every $\alpha \in [0, 1]$ that

$$\begin{aligned}
&\left\| H_{X^{-1/2}} \left[ X(\alpha)S(\alpha) - \mu(\alpha)I \right] \right\|_F \\
&\le (1-\alpha) \left\| H_{X^{-1/2}} (XS - \mu I) \right\|_F + \alpha \left\| H_{X^{-1/2}} \left( X\widetilde{\Delta S} + \widetilde{\Delta X}S \right) \right\|_F \\
&\quad + \alpha^2 \left\| H_{X^{-1/2}} (\Delta X \Delta S) \right\|_F \\
&\le (1-\alpha) \left\| X^{1/2}SX^{1/2} - \mu I \right\|_F + \alpha\, \tilde{\delta}_x \left\| X^{1/2}SX^{1/2} - \theta\mu I \right\| \\
&\quad + \alpha^2 \left\| X^{-1/2}\Delta X \Delta S X^{1/2} \right\|_F \\
&\le (1-\alpha) \left\| X^{1/2}SX^{1/2} - \mu I \right\|_F + \alpha\, \tilde{\delta}_x \left\| X^{1/2}SX^{1/2} - \theta\mu I \right\| + \alpha^2 \delta_x \delta_s.
\end{aligned}$$

□

The proof of next lemma is straightforward and therefore we omit the details.

**Lemma 4.** *If $(X, S) \in \mathcal{N}_F(\mu, \gamma)$ for some $\gamma \in (0, 1)$, then*

$$\|X^{1/2}S^{1/2}\|^2 \le (1+\gamma)\mu, \tag{11}$$

$$\|X^{-1/2}S^{-1/2}\|^2 \le [(1-\gamma)\mu]^{-1}, \tag{12}$$

$$\|X^{1/2}SX^{1/2} - \theta\mu I\|_F \le \left(\gamma + (1-\theta)\sqrt{n}\right)\mu, \quad \textit{for any } \theta \in [0,1], \tag{13}$$

$$(1-\gamma)n\mu \le X \bullet S \le (1+\gamma)n\mu. \tag{14}$$

The next result gives bounds on the quantities $\delta_x$, $\tilde{\delta}_x$ and $\delta_s$ defined in (10).

**Lemma 5.** *If $(X, S) \in \mathcal{N}_F(\mu, \gamma)$ for some $\gamma \in (0, 1)$, then*

$$\max\{\delta_x, \tilde{\delta}_x\} \le \frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma},$$

$$\delta_s \le \frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma}\mu,$$

*where $\delta_x$, $\tilde{\delta}_x$ and $\delta_s$ are defined in (10).*

*Proof.* Multiplying (4a) on the left by $X^{-1/2}$ and on the right by $S^{-1/2}$, squaring both sides of the resulting equation and noting the fact that $(\Delta X + \widetilde{\Delta X}) \bullet (\Delta S + \widetilde{\Delta S}) \ge 0$, we obtain

$$\begin{aligned}
&\left\| X^{-1/2}(\Delta X + \widetilde{\Delta X})S^{1/2} \right\|_F^2 + \left\| X^{1/2}(\Delta S + \widetilde{\Delta S})S^{-1/2} \right\|_F^2 \\
&\qquad \le \left\| X^{1/2}S^{1/2} - \sigma\mu X^{-1/2}S^{-1/2} \right\|_F^2 .
\end{aligned} \tag{15}$$

Using the fact that $\Delta X, \Delta S \in \mathcal{S}^n$, $\widetilde{\Delta X}, \widetilde{\Delta S} \in \mathcal{S}^n_\perp$ (and hence that $X^{1/2}\Delta X X^{1/2}$, $X^{1/2}\Delta S X^{1/2} \in \mathcal{S}^n$ and $X^{1/2}\widetilde{\Delta X} X^{1/2}$, $X^{1/2}\widetilde{\Delta S} X^{1/2} \in \mathcal{S}^n_\perp$), and

$$\frac{\|M + M^T\|_F}{2} \le \|M\|_F, \quad \frac{\|M - M^T\|_F}{2} \le \|M\|_F,$$

for any $M \in \Re^{n\times n}$, relations (10) and (15), and Lemma 4, we obtain

$$\begin{aligned}
\delta_s = \left\|X^{1/2}\Delta S X^{1/2}\right\|_F \;&\le\; \left\|X^{1/2}(\Delta S + \widetilde{\Delta S})X^{1/2}\right\|_F \\
&\le \left\|X^{1/2}(\Delta S + \widetilde{\Delta S})S^{-1/2}\right\|_F \left\|S^{1/2}X^{1/2}\right\| \\
&\le \left\|X^{1/2}S^{1/2} - \sigma\mu X^{-1/2}S^{-1/2}\right\|_F \left\|S^{1/2}X^{1/2}\right\| \\
&\le \left\|X^{1/2}SX^{1/2} - \sigma\mu I\right\|_F \left\|X^{-1/2}S^{-1/2}\right\| \left\|X^{1/2}S^{1/2}\right\| \\
&\le \left(\frac{1+\gamma}{1-\gamma}\right)^{1/2}\left(\gamma + (1-\sigma)\sqrt{n}\right)\mu \;\le\; \frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma}\mu,
\end{aligned}$$

and

$$\begin{aligned}
\max\{\delta_x, \tilde{\delta}_x\} &\le \max\left\{\left\|X^{-1/2}\Delta X X^{-1/2}\right\|_F, \left\|X^{-1/2}\widetilde{\Delta X} X^{-1/2}\right\|_F\right\} \\
&\le \left\|X^{-1/2}(\Delta X + \widetilde{\Delta X})X^{-1/2}\right\|_F \\
&\le \left\|X^{-1/2}(\Delta X + \widetilde{\Delta X})S^{1/2}\right\|_F \left\|S^{-1/2}X^{-1/2}\right\| \\
&\le \left\|X^{1/2}S^{1/2} - \sigma\mu X^{-1/2}S^{-1/2}\right\|_F \left\|X^{-1/2}S^{-1/2}\right\| \\
&\le \left\|X^{1/2}SX^{1/2} - \sigma\mu I\right\|_F \left\|X^{-1/2}S^{-1/2}\right\|^2 \\
&\le \frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma}.
\end{aligned}$$

□

Now we are ready to state the main result of this section.

**Lemma 6.** *Suppose that* $(X, S) \in \mathcal{N}_F(\mu, \gamma)$ *for some* $\gamma \in (0, 1)$ *and let* $(\Delta X, \widetilde{\Delta X}, \Delta S, \widetilde{\Delta S})$ *be the solution of (4). Then,*

$$\begin{aligned}
&\left\| H_{X^{-1/2}}\left[X(\alpha)S(\alpha) - \mu(\alpha)I\right]\right\|_F \\
&\le \left\{(1-\alpha)\gamma + \alpha\gamma\,\frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma} + \alpha^2\left(\frac{\gamma + (1-\sigma)\sqrt{n}}{1-\gamma}\right)^2\right\}\mu.
\end{aligned}$$

*Proof.* Follows immediately from (9) with $\theta = 1$, the assumption that $(X, S) \in \mathcal{N}_F(\mu, \gamma)$ and Lemma 5.

□

## 4. Algorithms

In this section, we establish polynomial iteration-complexity bounds for two primal-dual feasible interior-point algorithms for SDLCP based on the KSH family of search directions given by (4). Both algorithms are extensions of well-known algorithms for linear programming: the first one is a short-step path-following method which generalizes the algorithms presented in Kojima, Mizuno and Yoshise [28] and Monteiro and Adler [29, 30]; the second one is a predictor-corrector algorithm similar to the predictor-corrector LP method of Mizuno, Todd and Ye [31].

We start by stating two technical results. The first one is due to Monteiro (see Lemma 2.1 of [15]) and plays a crucial role in our analysis.

**Lemma 7.** *Suppose that* $(X, S) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ *and* $Q \in \Re^{n\times n}$ *is a nonsingular matrix. Then, for every* $\mu \in \Re$, *we have*

$$\|X^{1/2}SX^{1/2} - \mu I\|_F \leq \|H_Q(XS - \mu I)\|_F,$$

*with equality holding if* $QXSQ^{-1} \in \mathcal{S}^n$.

**Lemma 8.** *Let* $V, Q \in \Re^{n\times n}$ *be given. Suppose that* $Q$ *is nonsingular and that*

$$\|H_Q(V) - I\| < 1. \tag{16}$$

*Then, the matrix* $V$ *is nonsingular.*

*Proof.* Define $W \equiv QVQ^{-1}/2$. Condition (16) implies that $W + W^T \succ 0$, and this clearly implies that $W$ is nonsingular. Hence, $V$ is also nonsingular.

□

*4.1. Short-step path following algorithm*

In this subsection, we analyze the polynomial convergence of a short-step path following algorithm based on the KSH family of search directions.

We start by stating the algorithm that will be considered in this subsection.

**Algorithm-I:**

Choose constants $\gamma$ and $\delta$ in $(0, 1)$ satisfying the conditions of Theorem 2 below and let $\sigma \equiv 1 - \delta/\sqrt{n}$. Let $\mu_0 > 0$ and $(X^0, S^0) \in \mathcal{L}_{++}$ be such that $(X^0, S^0) \in \mathcal{N}_F(\mu_0, \gamma)$. Let $\varepsilon \in (0, 1)$.

**Repeat until** $\mu_k \leq \varepsilon\mu_0$, **do**

(1) Choose a linear subspace $\mathcal{L}^k_\perp$ satisfying [A3].

(2) Compute the solution $(\Delta X^k, \widetilde{\Delta X}^k, \Delta S^k, \widetilde{\Delta S}^k)$ of system (4) with $(X, S) = (X^k, S^k)$, $\mathcal{L}_\perp = \mathcal{L}^k_\perp$ and $\hat{\mu} = \sigma\mu_k$;

(3) Set $(X^{k+1}, S^{k+1}) \equiv (X^k, S^k) + (\Delta X^k, \Delta S^k)$ and $\mu_{k+1} = \sigma\mu_k$;

(4) Increment $k$ by 1.

**End**

When the constant $\Gamma$ defined in (17) is such that $\Gamma \leq \gamma$, the lemma below implies that the sequence $\{(X^k, S^k)\}$ generated by Algorithm-I is contained in the neighborhood $\mathcal{N}_F(\gamma)$. This lemma is also used in the analysis of the corrector (or centering) steps of the predictor-corrector algorithm presented in the next subsection.

**Lemma 9.** *Let $\gamma \in (0,1)$ and $\delta \in [0, n^{1/2})$ be constants satisfying*

$$\Gamma \equiv 2\left(\frac{\gamma+\delta}{1-\gamma}\right)^2 \left(1-\frac{\delta}{\sqrt{n}}\right)^{-1} < 1. \tag{17}$$

*Suppose that $(X,S) \in \mathcal{N}_F(\mu,\gamma)$ for some $\mu > 0$, and $(\Delta X, \widetilde{\Delta X}, \Delta S, \widetilde{\Delta S})$ is the solution of system (4) with $\hat{\mu} = \sigma\mu$ and $\sigma = 1 - \delta/\sqrt{n}$. Then, $(X+\Delta X, S+\Delta S) \in \mathcal{N}_F(\sigma\mu, \Gamma)$.*

*Proof.* It follows from Lemma 6, the definition of $\sigma$ and (17) that for every $\alpha \in [0,1]$,

$$\begin{aligned}
&\left\| H_{X^{-1/2}}\left[X(\alpha)S(\alpha) - \mu(\alpha)I\right]\right\|_F \\
&\le \left\{(1-\alpha)\,\gamma + \alpha\,\gamma\,\frac{\gamma+(1-\sigma)\sqrt{n}}{1-\gamma} + \alpha^2\left(\frac{\gamma+(1-\sigma)\sqrt{n}}{1-\gamma}\right)^2\right\}\mu \\
&= (1-\alpha)\gamma\mu + \left\{\alpha\,\gamma\,\frac{\gamma+\delta}{1-\gamma} + \alpha^2\left(\frac{\gamma+\delta}{1-\gamma}\right)^2\right\}\mu \\
&\le (1-\alpha)\gamma\mu + 2\alpha\left(\frac{\gamma+\delta}{1-\gamma}\right)^2\mu \;=\; (1-\alpha)\gamma\mu + \alpha\Gamma\left(1-\frac{\delta}{\sqrt{n}}\right)\mu \\
&= \{(1-\alpha)\gamma + \sigma\Gamma\alpha\}\,\mu,
\end{aligned}$$

and hence, in view of (7) and (17), we have

$$\left\| H_{X^{-1/2}}\left[\frac{X(\alpha)S(\alpha)}{\mu(\alpha)}\right] - I\right\|_F \le \frac{(1-\alpha)\gamma+\sigma\Gamma\alpha}{1-\alpha+\sigma\alpha} \le \max\{\gamma,\Gamma\} < 1.$$

By Lemma 8, this implies that $X(\alpha)S(\alpha)$ is nonsingular for every $\alpha \in (0,1]$. Hence, $X(\alpha)$ and $S(\alpha)$ are also nonsingular for every $\alpha \in (0,1]$. Using the fact that $(X,S) \in \mathcal{L}_{++}$, $(X+\Delta X, S+\Delta S) \in \mathcal{L}$ and a simple continuity argument, we see $(X(\alpha), S(\alpha)) \in \mathcal{L}_{++} \subseteq \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ for every $\alpha \in (0,1]$. Applying Lemma 7 with $(X,S) = (X(\alpha), S(\alpha))$ and $Q = X^{-1/2}$, we conclude that for every $\alpha \in [0,1]$,

$$\begin{aligned}
\|X(\alpha)^{1/2}S(\alpha)X(\alpha)^{1/2} - \mu(\alpha)I\|_F &\le \|H_{X^{-1/2}}(X(\alpha)S(\alpha)-\mu(\alpha)I)\|_F \\
&\le \|X^{-1/2}X(\alpha)S(\alpha)X^{1/2} - \mu(\alpha)I\|_F \\
&\le \{(1-\alpha)\gamma + \sigma\Gamma\alpha\}\,\mu.
\end{aligned}$$

Setting $\alpha = 1$ in the last relation and using the fact that $(X(1), S(1)) \in \mathcal{L}_{++}$ together with (6) and (7), we conclude that $(X(1), S(1)) \equiv (X+\Delta X, S+\Delta S) \in \mathcal{N}_F(\sigma\mu, \Gamma)$. □

As an immediate consequence of Lemma 9, we have the following convergence result for Algorithm-I.

**Theorem 2.** *Suppose that $\gamma$ and $\delta$ are constants in $(0,1)$ such that $\Gamma$ defined by (17) satisfies $\Gamma \le \gamma$. Then, every iterate $(X^k, S^k)$ generated by Algorithm-I is in $\mathcal{N}_F(\mu_k,\gamma) \subseteq \mathcal{N}_F(\gamma)$ and satisfies*

$$X^k \bullet S^k \le \frac{1+\gamma}{1-\gamma}\left(1-\frac{\delta}{\sqrt{n}}\right)^k (X^0 \bullet S^0). \tag{18}$$

*Moreover, Algorithm-I terminates in at most $\mathcal{O}(\sqrt{n}\log\varepsilon^{-1})$ iterations.*

*Proof.* The proof that every iterate $(X^k, S^k)$ is in $\mathcal{N}_F(\mu_k, \gamma)$ follows immediately from Lemma 9 and a simple induction argument. Relation (18) follows from the fact that $\mu_k = \sigma^k \mu_0$ and relation (14).

□

Examples of constants $\gamma$ and $\delta$ satisfying the conditions of Theorem 2 are $\gamma = \delta = 1/25$.

### *4.2. Predictor-corrector algorithm*

In this subsection, we give the polynomial convergence analysis of a predictor-corrector algorithm which is a direct extension of the LP predictor-corrector algorithm studied by Mizuno, Todd and Ye [31].

The algorithm considered in this subsection is as follows.

**Algorithm-II:**

Choose a constant $0 < \tau < 1/2$ satisfying the conditions of Theorem 3 below.
Let $\varepsilon \in (0, 1)$ and $(X^0, S^0) \in \mathcal{L}_{++}$ be such that $(X^0, S^0) \in \mathcal{N}_F(\mu_0, \tau)$,

**Repeat until** $\mu_k \le \varepsilon\mu_0$, **do**

(1) Choose a linear subspace $\mathcal{L}_\perp^k$ satisfying [A3];
(2) Compute the solution $(\Delta X_P^k, \widetilde{\Delta X}_P^k, \Delta S_P^k, \widetilde{\Delta S}_P^k)$ of system (4) with $(X, S) = (X^k, S^k)$, $\mathcal{L}_\perp = \mathcal{L}_\perp^k$ and $\hat{\mu} = 0$;
(3) Let $\alpha_k \equiv \max\{\alpha \in [0, 1] : (X^k(\alpha'), S^k(\alpha')) \in \mathcal{N}_F((1-\alpha')\mu_k, 2\tau)$, $\forall \alpha' \in [0, \alpha]\}$, where $(X^k(\alpha), S^k(\alpha)) \equiv (X^k + \alpha\Delta X_P^k, S^k + \alpha\Delta S_P^k)$;
(4) Let $(\widehat{X}^k, \widehat{S}^k) \equiv (X^k, S^k) + \alpha_k(\Delta X_P^k, \Delta S_P^k)$ and $\mu_{k+1} \equiv (1-\alpha_k)\mu_k$;
(5) Choose a linear subspace $\hat{\mathcal{L}}_\perp^k$ satisfying [A3];
(6) Compute the solution $(\Delta X_C^k, \widetilde{\Delta X}_C^k, \Delta S_C^k, \widetilde{\Delta S}_C^k)$ of system (4) with $(X, S) = (\widehat{X}^k, \widehat{S}^k)$, $\hat{\mu} = \mu_{k+1}$ and $\mathcal{L}_\perp = \hat{\mathcal{L}}_\perp^k$;
(7) Set $(X^{k+1}, S^{k+1}) \equiv (\widehat{X}^k, \widehat{S}^k) + (\Delta X_C^k, \Delta S_C^k)$;
(8) Increment $k$ by 1.

**End**

The following result provides the polynomial convergence analysis of the above algorithm.

**Theorem 3.** *Assume that* $\tau \in (0, 1/30]$. *Then, Algorithm-II satisfies the following statements:*

a) *for every* $k \ge 0$, $(X^k, S^k) \in \mathcal{N}_F(\tau)$ *and* $(\widehat{X}^k, \widehat{S}^k) \in \mathcal{N}_F(2\tau)$*;*
b) *for every* $k \ge 0$, $X^k \bullet S^k \le \dfrac{1+\tau}{1-\tau}(1-\bar{\alpha})^k X^0 \bullet S^0$, *where* $\bar{\alpha} = 1/\mathcal{O}(\sqrt{n})$*;*
c) *the algorithm terminates in at most* $\mathcal{O}(\sqrt{n}\log \varepsilon^{-1})$ *iterations.*

*Proof.* Statement (c) and the well-definedness of Algorithm-II follow directly from (a) and (b). In turn, these two statements follow by a simple induction argument, the two lemmas below and relation (14).

□

The following lemma analyzes the predictor step of Algorithm-II, namely the step described in items (1)-(4) of Algorithm-II.

**Lemma 10.** *Suppose that $(X, S) \in \mathcal{N}_F(\mu, \tau)$ for some $\tau \in (0, 1/2)$. For some subspace $\mathcal{L}_\perp$ satisfying [A3], let $(\Delta X_P, \widetilde{\Delta X}_P, \Delta S_P, \widetilde{\Delta S}_P)$ denote the solution of (4) with $\hat{\mu} = 0$. Let $\bar{\alpha}$ denote the unique positive root of the second-order polynomial $p(\alpha)$ defined as*

$$p(\alpha) = \left(\frac{\tau + \sqrt{n}}{1-\tau}\right)^2 \alpha^2 + \tau \left[\left(\frac{\tau + \sqrt{n}}{1-\tau}\right) + 1\right] \alpha - \tau \tag{19}$$

*Then, for any $\alpha \in [0, \bar{\alpha}]$, we have:*

$$(X(\alpha), S(\alpha)) \equiv (X + \alpha \Delta X_P, S + \alpha \Delta S_P) \in \mathcal{N}_F((1-\alpha)\mu, 2\tau). \tag{20}$$

*Moreover, $\bar{\alpha} = 1/\mathcal{O}(n^{1/2})$.*

*Proof.* Using Lemma 6 with $\gamma = \tau$ and $\sigma = 0$, the fact that $p(\alpha) \leq 0$ for $\alpha \in [0, \bar{\alpha}]$, $\tau < 1/2$ and (19), we obtain

$$\begin{aligned}\left\| H_{X^{-1/2}}[X(\alpha)S(\alpha) - \mu(\alpha)]\right\|_F &\leq \left\{(1-\alpha)\tau + \tau\left(\frac{\tau+\sqrt{n}}{1-\tau}\right)\alpha + \left(\frac{\tau+\sqrt{n}}{1-\tau}\right)^2 \alpha^2\right\}\mu \\ &= 2\tau\mu(\alpha) + p(\alpha)\mu \ \leq \ 2\tau\mu(\alpha).\end{aligned}$$

An argument similar to the one used in Lemma 9 together with (7) and the fact that $2\tau < 1$ and $\hat{\mu} = 0$ (or equivalently, $\sigma = 0$) can be used to show that (20) holds. The assertion that $\bar{\alpha} = 1/\mathcal{O}(n^{1/2})$ follows by a straightforward verification.

□

The following lemma analyzes the corrector step of Algorithm-II, namely the step described in items (5)-(7) of Algorithm-II.

**Lemma 11.** *Suppose $(\widehat{X}, \widehat{S})$ is in $\mathcal{N}_F(\mu, 2\tau)$ for some $\tau \in (0, 1/30]$. Let $(\Delta X_C, \widetilde{\Delta X}_C, \Delta S_C, \widetilde{\Delta S}_C)$ denote the solution of (4) with $(X, S) = (\widehat{X}, \widehat{S})$, $\hat{\mu} = \mu$ and $\mathcal{L}_\perp$ satisfying [A3]. Then,*

$$(\widehat{X}, \widehat{S}) + (\Delta X_C, \Delta S_C) \in \mathcal{N}_F(\mu, \tau).$$

*Proof.* Follows immediately from Lemma 9 with $\sigma = 1$ (or equivalently, $\delta = 0$), $(X, S) = (\widehat{X}, \widehat{S})$ and $\gamma = 2\tau$, and noting that $\Gamma$ defined by (17) satisfies $\Gamma \leq \tau$ when $\tau \leq 1/30$.

□

## 5. Concluding remarks

For simplicity, we have analyzed two algorithms whose sequence $\{\mu_k\}$ in general differs from the sequence of normalized complementarity gaps $\{(X^k \bullet S^k)/n\}$. At the expense of a slightly more complicated analysis, it is possible to develop algorithms similar to the ones presented here in which $\mu_k = (X^k \bullet S^k)/n$ for every $k$.

The algorithms of this paper are based on the Frobenius neighborhood $\mathcal{N}_F(\gamma)$ of the central path. An interesting topic for future research would be to establish polynomial convergence of algorithms based on the KSH family of search directions which use one of the following two wider neighborhoods of the central path:

$$\left\{(X, S) \in \mathcal{L}_+ : \|X^{1/2}SX^{1/2} - \mu I\| \leq \gamma\mu, \text{ for some } \mu > 0\right\},$$
$$\{(X, S) \in \mathcal{L}_+ : \lambda_{\min}(XS - \mu I) \geq -\gamma\mu, \text{ for some } \mu > 0\}.$$

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