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A Geometric View of Parametric Linear Programming[1]

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Abstract. We present a new definition of optimality intervals for the parametric right-hand side linear programming (parametric RHS LP) Problem $\varphi(\lambda) = \min\{c^T x \mid Ax = b + \lambda\bar{b},\ x \geq 0\}$. We then show that an optimality interval consists either of a breakpoint or the open interval between two consecutive breakpoints of the continuous piecewise linear convex function $\varphi(\lambda)$. As a consequence, the optimality intervals form a partition of the closed interval $\{\lambda;\ |\varphi(\lambda)| < \infty\}$. Based on these optimality intervals, we also introduce an algorithm for solving the parametric RHS LP problem which requires an LP solver as a subroutine. If a polynomial-time LP solver is used to implement this subroutine, we obtain a substantial improvement on the complexity of those parametric RHS LP instances which exhibit degeneracy. When the number of breakpoints of $\varphi(\lambda)$ is polynomial in terms of the size of the parametric problem, we show that the latter can be solved in polynomial time.

Key Words. Parametric linear programming, Sensitivity analysis, Postoptimality analysis, Linear programming.

1. Introduction. The subject of this paper is to study the parametric right-hand side linear programming (parametric RHS LP) problem as follows:

$$(P_\lambda) \qquad \min\{c^T x \mid Ax = b + \lambda\bar{b},\ x \geq 0\},$$

where A is an $m \times n$ matrix and b, $\bar{b}$, and c are vectors of dimensions m, m, and n, respectively. The parametric RHS LP problem (P_λ), $\lambda \in \mathbf{R}$, consists of solving each linear programming (LP) problem (P_λ) for all values of $\lambda \in \mathbf{R}$ (or for λ in a certain required interval). If $\varphi(\lambda)$ denotes the optimal value of (P_λ), it is well known that the function $\lambda \in \mathbf{R} \to \varphi(\lambda)$ is a convex piecewise linear continuous function. In view of this property, only a finite amount of information is necessary to solve the parametric RHS LP problem. Basically, it consists of finding the "breakpoints" of $\varphi(\lambda)$ and an optimal solution of (P_λ) for all breakpoints λ.

We present a way of approaching this problem which differs from the usual method based on the simplex method. Our main motivation to look back into this problem was the introduction of new methods for solving LP problems like the ellipsoid method introduced by Khachiyan [7] and the new interior point algorithm presented by Karmarkar [6].

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The existing method to solve this problem is the parametric RHS LP simplex method which was first discussed by Gass and Saaty [5] a few years after the simplex method was developed by Dantzig. Many textbooks describe this variant of the simplex method. See, for instance, Dantzig [1] and Murty [9]. The theory of sensitivity and parametric analysis both in discrete and continuous linear (and nonlinear) optimization has been the subject of intensive research. For example, the book by Gal [4] contains about 700 references related to sensitivity and parametric analysis.

Both the existing theory of sensitivity and parametric analysis depends crucially on the concept of the optimality (or characteristic) interval associated with an optimal basis, that is, the set of values of λ for which this basis is optimal for the LP problem (P_λ).

In this paper we introduce a different definition of optimality intervals and derive an algorithm for solving the parametric RHS LP problem which can be implemented with the aid of any LP solver. As a first step we have to get rid of the concept of basis and introduce another invariant associated with the problem in order to define our optimality intervals. This is done by considering those partitions (B, N), which we call optimal partitions, such that $B \cup N = \{1, \dots, n\}$, $B \cap N = \emptyset$ and $(x_j \geq 0, j \in N)$ and $(A_{.j}^T y \leq c_j, j \in B)$ are, respectively, the set of always-active constraints with respect to the primal optimal face and the dual optimal face of some problem (P_λ). We then show that an optimality interval is either an open interval between two consecutive breakpoints of $\varphi(\lambda)$ or consists of a breakpoint itself. This shows that the real line is covered in a unique way using these optimality intervals. This is in contrast with the basis optimality intervals where even the closed interval determined by two consecutive breakpoints of $\varphi(\lambda)$ can be covered in many ways with possibly an exponential number of these intervals.

The second step is to provide an algorithm for the parametric RHS LP problem, based on any LP solver, that computes a sequence of optimal partitions and their associated optimality intervals so that at the end we have covered the required interval by these optimality intervals. The approach is the same as in the existing pivot method which successively finds adjacent optimality intervals either going to the left or to the right of the real line. However, our approach solves an LP problem to find the adjacent partition and the corresponding optimality interval.

It is well known that the parametric RHS LP problem cannot be solved in polynomial time due the existence of instances of the problem whose corresponding function $\varphi(\lambda)$ exhibits an exponential number of breakpoints. One of the main consequences of our algorithm is an affirmative answer to the following related computational complexity issue: Can the parametric RHS LP problem be solved in time polynomially bounded by the size of the input and the number of breakpoints of $\varphi(\lambda)$? For nondegenerate problems, the answer to this issue is rather trivial and is provided by the parametric RHS simplex method discussed above. For degenerate problems, we show in Section 4 that the parametric RHS LP problem can be solved in $O(kZ)$ where k is the number of breakpoints and Z is the complexity of solving a single LP problem of the same dimension.

Our paper is organized as follows. In Section 2 we introduce some notation and review basic results related to our development in this paper. In Section 3 we define our optimality intervals and provide a complete characterization of these intervals. In Section 4 we describe the algorithm for the parametric RHS problem based on any LP solver. In Section 5 we conclude the paper with some remarks.

2. Problem Description and Some Theoretical Background. In this section we introduce the class of problems with which this paper is concerned. We also review some results pertinent to the present work.

2.1. Preliminary Notations. $\mathbf{R}^n$, $\mathbf{R}^n_+$, and $\mathbf{R}^n_{++}$ denotes the sets of n-vectors with reals components, nonnegative reals components, and (strictly) positive reals components, respectively. Let A be an $m \times n$ matrix. Given a subset B of the index set $\{1, \ldots, n\}$, we denote by A_B the submatrix of A associated with the index set B. Also, the subspace generated by the columns of A is denoted by *range*(A). The ith row and jth column of A is denoted by $A_{i\cdot}$ and $A_{\cdot j}$, respectively. A closed interval in $\mathbf{R}$ with extreme points α and β is denoted by $[\alpha, \beta]$ even when either $\alpha = -\infty$ or $\beta = \infty$. The closure of a set $X \subseteq \mathbf{R}^n$ is denoted by cl X. $\bar{\mathbf{R}}$ denotes the set of extended reals, that is, $\bar{\mathbf{R}} \equiv \mathbf{R} \cup \{-\infty, \infty\}$.

2.2. True Inequalities. Consider a polyhedron $Q \subseteq \mathbf{R}^n$, that is, $Q \equiv \{x \in \mathbf{R}^n | Ax \leq b; Cx = d\}$. The system of linear constraints $Ax \leq b$, $Cx = d$ is then said to be a representation of Q. A subset F of Q is a face of Q if either $F = \varnothing$ or F is the set of optimal solutions for $\min\{c^T x | x \in Q\}$ for some $c \in \mathbf{R}^n$. We say that an inequality $ax \leq \beta$ of the system $Ax \leq b$ is a *true inequality for the face F if $ax < \beta$ for some $x \in F$.* When the face F is the whole polyhedron $\{x | Ax \leq b, Cx = d\}$, the set of true inequalities for the face F is referred to simply as the set of *true inequalities for the system $Ax \leq b$, $Cx = d$.* Let $A'x \leq b'$ be the true inequalities for the face F and $A''x \leq b''$ be the other inequalities from $Ax \leq b$. It can be shown (see, for example, Section 8.3 of [10]) that

$$F = \{x \in \mathbf{R}^n | A'x \leq b';\ A''x = b'';\ Cx = d\}. \tag{2.1}$$

The following observation easily follows from (2.1) and the definition of true inequalities.

REMARK 2.1. Let F and F' be two faces of the polyhedron Q. Then $F \subseteq F'$ if and only if every true inequality for F is also a true inequality for F'.

The following result will be useful later and can be easily proved.

PROPOSITION 2.1. *Let the face F and the matrices A', A'', and C be as above and let $f \in \mathbf{R}^n$ be given. Then the linear function $x \to f^T x$ is constant on F if and only if $f \in range[(A'')^T, C^T]$.*

2.3. Optimal Partitions. Consider the LP problem in standard form

$$(P) \qquad \min\{c^T x \mid Ax = b;\ x \geq 0\}$$

and its dual

$$(D) \qquad \max\{b^T y \mid A^T y \leq c\},$$

where A is an $m \times n$ matrix, c is an n-vector, and b is an m-vector. Let X^* and Y^* denote the set of optimal solutions of problems (P) and (D), respectively. Assume that X^* (and consequently Y^*) is nonempty. Clearly, X^* (resp. Y^*) is a face of the polyhedron of feasible solutions for problem (P) (resp. (D)). Let the inequalities $x_j \geq 0$ with $j \in B \subseteq \{1, \ldots, n\}$ and the inequalities $A_{\cdot j}^T y \leq c_j$ with $j \in N \subseteq \{1, \ldots, n\}$ be the set of true inequalities for the faces X^* and Y^*, respectively. This is equivalent to saying that

$$(2.2) \qquad B = \{j \mid x_j > 0 \text{ for some } x \in X^* \text{ and } j = 1, \ldots, n\},$$

$$(2.3) \qquad N = \{j \mid c_j - A_{\cdot j}^T y > 0 \text{ for some } y \in Y^* \text{ and } j = 1, \ldots, n\}.$$

We then have the following well-known result.

Proposition 2.2. *Assume that X^* is nonempty and let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ be as in* (2.2) *and* (2.3). *Then $B \cap N = \varnothing$ and $B \cup N = \{1, \ldots, n\}$.*

The pair of index sets (B, N) then determines a partition of the index set $\{1, \ldots, n\}$. We refer to (B, N) as the optimal partition associated with problem (P). For a proof of Proposition 2.2, see, for example, Section 7.9 of [10]. In terms of the partition (B, N), the optimal faces X^* and Y^* can be written as

$$X^* = \{x \mid A_B x_B = b;\ x_B \geq 0;\ x_N = 0\},$$
$$Y^* = \{y \mid A_B^T y = c_B;\ A_N^T y \leq c_N\}.$$

2.4. Description of the Parametric RHS LP Problem and Related Preliminary Results. In this subsection we introduce the problem which is the object of our analysis in this paper.

Consider the parametrized family of LP problems in standard form

$$(P_\lambda) \qquad \min\{c^T x \mid Ax = b + \lambda \bar{b};\ x \geq 0\}$$

and the corresponding parametrized family of dual problems

$$(D_\lambda) \qquad \max\{(b + \bar{b})^T y \mid A^T y \leq c\},$$

where b, $\bar{b}$ are m-vectors, c is an n-vector, A is an $m \times n$ matrix, and $\lambda \in \mathbf{R}$. Solving (P_λ) for all $\lambda \in \mathbf{R}$ is known as the parametric RHS LP problem. We denote the optimal value of (P_λ) by $\varphi(\lambda)$ with the convention that $\varphi(\lambda) = \infty$ if (P_λ) is infeasible and $\varphi(\lambda) = -\infty$ if (P_λ) is feasible and unbounded. With this convention, we then obtain an extended convex function $\varphi: \mathbf{R} \to \bar{\mathbf{R}}$, that is, a function taking values on $\bar{\mathbf{R}}$ and whose epigraph epi $\varphi \equiv \{(x, \theta) \in \mathbf{R}^n \times \mathbf{R} \mid \theta \geq \varphi(x)\}$ is a convex set (see Proposition 2.3 below).

The next proposition characterizes the "shape" of the function φ.

PROPOSITION 2.3. *There exists a closed interval* $[\alpha, \beta]$ *(possibly empty) such that:*

(a) $\varphi(\lambda) = \infty$ *for all* $\lambda \notin [\alpha, \beta]$.
(b) *Either* $\varphi(\lambda) = -\infty$ *for all* $\lambda \in [\alpha, \beta]$ *or* $\varphi([\alpha, \beta]) \subseteq \mathbf{R}$ *and in this case* $\varphi \mid [\alpha, \beta]$ *is a continuous convex piecewise linear function.*

For a proof of Proposition 2.3, see, for example, Theorem 8.3, p. 288, of [9]. The cases in which either $[\alpha, \beta]$ is empty or $\varphi(\lambda) = -\infty$ for all $\lambda \in [\alpha, \beta]$ present no difficulty to our analysis. Henceforth, we make the following assumption.

ASSUMPTION 2.1. $[\alpha, \beta]$ *is nonempty and* $\varphi(\lambda) \in \mathbf{R}$ *for all* $\lambda \in [\alpha, \beta]$.

In view of Assumption 2.1, there exists a finite set of points $\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \beta$ and real constants g_i, h_i, $i \in \{1, \ldots, k\}$, such that $\varphi(\lambda) = g_i \lambda + h_i$ for all $\lambda \in [\lambda_{i-1}, \lambda_i]$. The convexity of φ implies that $g_1 < g_2 < \cdots < g_k$. Obviously, φ has left and right derivatives for all $\lambda \in [\alpha, \beta]$. Indeed, for all $i \in \{1, \ldots, k\}$, $\varphi'(\lambda) = g_i$ if $\lambda \in (\lambda_{i-1}, \lambda_i)$, $\varphi'_+(\lambda_{i-1}) = g_i$, and $\varphi'_-(\lambda_i) = g_i$. By convention, if α (resp. β) is finite, we let $\varphi'_-(\alpha) = -\infty$ (resp. $\varphi'_+(\beta) = \infty$).

Throughout this paper we let $X(\lambda)$ and $Y(\lambda)$, where $\lambda \in [\alpha, \beta]$, denote the primal and dual optimal faces for problems (P_λ) and (D_λ), respectively. Also $(B(\lambda), N(\lambda))$ denotes the optimal partition associated with problem (P_λ) where $\lambda \in [\alpha, \beta]$.

The next proposition expresses the left and right derivatives of the function φ in terms of certain LP problems.

PROPOSITION 2.4. *For any* $\lambda \in [\alpha, \beta]$, *the left and right derivatives* $\varphi'_-(\lambda)$ *and* $\varphi'_+(\lambda)$ *are given by*

$$\varphi'_-(\lambda) = \min\{\bar{b}^T y \mid y \in Y(\lambda)\},$$

$$\varphi'_+(\lambda) = \max\{\bar{b}^T y \mid y \in Y(\lambda)\},$$

For a proof of Proposition 2.4, see, for example, Theorem 8.2, p. 288, of [9].

3. Characterization of Optimality Sets of Optimal Partitions. The existing theory of sensitivity and parametric analysis depends crucially on the concept of the optimality (or characteristic) interval associated with an optimal basis, that is, a basis which is primal and dual feasible for some problem (P_λ) where $\lambda \in [\alpha, \beta]$. In this case, the optimality set of an optimal basis is defined to be the set of all $\lambda \in [\alpha, \beta]$ for which such basis is optimal for the LP problem (P_λ). Hence, this theory studies the invariance of the optimality of a basis with respect to change on the parameter λ. Instead, our analysis is based on the optimality set of an optimal partition. In this section we introduce and characterize the optimality set of an optimal partition. We also present some results relating "adjacent" optimal partitions.

We start by defining the optimality set of an optimal partition. Let $(B, N) \equiv (B(\lambda^*), N(\lambda^*))$ be an optimal partition associated with (P_{λ^*}) where $\lambda^* \in [\alpha, \beta]$. The optimality set of (B, N) is the set defined as

$$\Lambda(B, N) \equiv \{\lambda \mid (B(\lambda), N(\lambda)) = (B, N)\}.$$

We also consider the set associated with the optimal partition (B, N) as follows:

$$\begin{aligned}\bar{\Lambda}(B, N) &\equiv \{\lambda \mid B(\lambda) \subseteq B\} \\ &= \{\lambda \mid N(\lambda) \supseteq N\}.\end{aligned}$$

Note that $\Lambda(B, N) \subseteq \bar{\Lambda}(B, N)$. Note also that there are a finite number of optimality sets, one for each distinct optimal partition that appears in $[\alpha, \beta]$. Moreover, these optimality sets form a partition of $[\alpha, \beta]$. Using Remark 2.1, we can easily present equivalent definitions of the sets $\Lambda(B, N)$ and $\bar{\Lambda}(B, N)$ in terms of the dual optimal faces $Y(\lambda)$ as follows:

$$\begin{aligned}\Lambda(B, N) &\equiv \{\lambda \mid Y(\lambda) = Y(\lambda^*)\}, \\ \bar{\Lambda}(B, N) &\equiv \{\lambda \mid Y(\lambda) \supseteq Y(\lambda^*)\},\end{aligned} \tag{3.1}$$

where we recall that λ^* is such that $(B, N) = (B(\lambda^*), N(\lambda^*))$.

Our main result, the characterization of the optimality sets, is given in the following theorem.

THEOREM 3.1. *Let $(B, N) = (B(\lambda^*), N(\lambda^*))$ for $\lambda^* \in [\alpha, \beta]$. Then:*

(a) *If $\lambda^* = \lambda_i$ for some $i \in \{0, 1, \ldots, k\}$, that is, λ^* is a breakpoint of $\varphi(\lambda)$, then $\Lambda(B, N) = \bar{\Lambda}(B, N) = \{\lambda^*\}$.*
(b) *If $\lambda^* \in (\lambda_{i-1}, \lambda_i)$ for some $i \in \{1, \ldots, k\}$, then $\Lambda(B, N) = (\lambda_{i-1}, \lambda_i)$ and $\bar{\Lambda}(B, N) = [\lambda_{i-1}, \lambda_i]$.*

In order to prove Theorem 3.1, we proceed as follows. Consider an optimal partition $(B, N) = (B(\lambda), N(\lambda))$ for some $\lambda \in [\alpha, \beta]$. First, we show that the optimality sets $\Lambda(B, N)$ and $\bar{\Lambda}(B, N)$ can be expressed as linear projections of certain polyhedral sets (Lemma 3.1). This fact immediately implies that $\Lambda(B, N)$ and $\bar{\Lambda}(B, N)$ are intervals which, in Lemma 3.2, are characterized by a certain algebraic condition on A_B and $\bar{b}$. In view of this result, from now on we refer to the optimality sets $\Lambda(B, N)$ as optimality intervals. Finally, we show in Lemma 3.3 that this algebraic condition is related to the condition that λ be a breakpoint, Theorem 3.1 then follows from these three lemmas.

We now introduce new sets as follows. For $B \subseteq \{1, \ldots, n\}$ and $N = \{1, \ldots, n\} - B$, let

$$X_B(\lambda) = \{x \in \mathbf{R}^n \mid Ax = b + \lambda\bar{b};\ x_B > 0;\ x_N = 0\},$$
$$\bar{X}_B(\lambda) = \{x \in \mathbf{R}^n \mid Ax = b + \lambda\bar{b};\ x_B \geq 0;\ x_N = 0\},$$
$$\Gamma(B) = \{\lambda \mid X_B(\lambda) \neq \varnothing\},$$
$$\bar{\Gamma}(B) = \{\lambda \mid \bar{X}_B(\lambda) \neq \varnothing\},$$
$$Y_B = \{y \in \mathbf{R}^m \mid A_B^T y = c_B;\ A_N^T y < c_N\}.$$

An equivalent definition of the sets $\Gamma(B)$ and $\bar{\Gamma}(B)$ which we use later is as follows. Let $\pi\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ denote the projection $\pi(\lambda, x) = \lambda$, $(\lambda, x) \in \mathbf{R} \times \mathbf{R}^n$ and consider the sets Q_B and $\bar{Q}_B$ as follows:

$$Q_B = \{\lambda, x) \in \mathbf{R} \times \mathbf{R}^n \mid A_B x_B - \lambda\bar{b} = b,\ x_B > 0,\ x_N = 0\},$$
$$\bar{Q}_B = \{(\lambda, x) \in \mathbf{R} \times \mathbf{R}^n \mid A_B x_B - \lambda\bar{b} = b,\ x_B \geq 0,\ x_N = 0\}.$$

Then the sets $\Gamma(B)$ and $\bar{\Gamma}(B)$ are the projection sets $\pi(Q_B)$ and $\pi(\bar{Q}_B)$, respectively. The following lemma relates the projection sets $\Gamma(B)$ and $\bar{\Gamma}(B)$ to the sets $\Lambda(B, N)$ and $\bar{\Lambda}(B, N)$, $N = \{1, \ldots, n\} - B$, when (B, N) is an optimal partition.

Lemma 3.1. *Let $(B, N) = (B(\lambda^*), N(\lambda^*))$ for some $\lambda^* \in [\alpha, \beta]$. Then:*

(a) $\Lambda(B, N) = \Gamma(B)$.
(b) $\bar{\Lambda}(B, N) = \bar{\Gamma}(B)$.
(c) $X(\lambda) = \bar{X}_B(\lambda)$ *for all* $\lambda \in \bar{\Gamma}(B)$ (*and hence for all* $\lambda \in \bar{\Lambda}(B, N)$).

Proof. Note first that Proposition 2.2 easily implies that $(B, N) = (B(\lambda), N(\lambda))$ if and only if $X_B(\lambda) \neq \varnothing$ and $Y_B \neq \varnothing$. Clearly, $Y_B \neq \varnothing$ since $(B, N) = (B(\lambda^*), N(\lambda^*))$ by assumption. These two observations imply (a). We next show (b) and (c). By complementary slackness condition, $\bar{X}_B(\lambda) \neq \varnothing$ implies $Y_B \subseteq Y(\lambda)$ which implies that $X(\lambda) = \bar{X}_B(\lambda)$. This shows (c). Clearly, (c) implies that $B(\lambda) \subseteq B$ for all $\lambda \in \bar{\Gamma}(B)$. Hence, $\bar{\Gamma}(B) \subseteq \bar{\Lambda}(B, N)$. The inclusion $\bar{\Gamma}(B) \supseteq \bar{\Lambda}(B, N)$ is immediate since $B(\lambda) \subseteq B$ implies $\varnothing \neq X(\lambda) \equiv \bar{X}_{B(\lambda)}(\lambda) \subseteq \bar{X}_B(\lambda)$. Hence (b) follows. □

Lemma 3.2. *Let $B \subseteq \{1, \ldots, n\}$ be given. Assume that $\Gamma(B) \neq \varnothing$ and let $\lambda^* \in \Gamma(B)$. Then:*

(a) $\Gamma(B) = \{\lambda^*\}$ *if and only if* $\bar{b} \notin \text{range}(A_B)$.
(b) $\Gamma(B)$ *is an open interval* (*containing* λ^*) *if and only if* $\bar{b} \in \text{range}(A_B)$.
(c) $\bar{\Gamma}(B) = \text{cl}(\Gamma(B))$.

Proof. Consider the sets Q_B and $\bar{Q}_B$ and the projection $\pi(\lambda, x) = x$ mentioned before the statement of Lemma 3.1. We know that $\pi(Q_B) = \Gamma(B)$ and $\pi(\bar{Q}_B) = \bar{\Gamma}(B)$. Since both Q_B and $\bar{Q}_B$ are convex sets and π is linear, it follows that both $\Gamma(B)$ and $\bar{\Gamma}(B)$ are convex sets and hence intervals. We first show (c). Clearly, since

$Q_B \neq \emptyset$, we have $\bar{Q}_B = \text{cl } Q_B$. Since π is continuous, this implies that $\bar{\Gamma}(B) = \pi(\bar{Q}_B) \subseteq \text{cl } \pi(Q_B) = \text{cl } \Gamma(B)$. Next, observe that $\bar{\Gamma}(B)$ is closed since $\bar{Q}_B$ is a polyhedron and $\pi(\bar{Q}_B) = \bar{\Gamma}(B)$. Since $\Gamma(B) \subseteq \bar{\Gamma}(B)$, it follows that $\text{cl } \Gamma(B) \subseteq \bar{\Gamma}(B)$. Hence (c) follows. We next show (a) and (b). Assume first that $\bar{b} \notin \text{range}(A_B)$. We will show that $\Gamma(B) = \{\lambda^*\}$. Indeed, since $\lambda^* \in \Gamma(B)$, we have $b + \lambda^* \bar{b} \in \text{range}(A_B)$. Since $\bar{b} \notin \text{range}(A_B)$, we can easily see that $b + \lambda \bar{b} \notin \text{range}(A_B)$ for any $\lambda \neq \lambda^*$. Therefore, $X_B(\lambda) = \emptyset$ for any $\lambda \neq \lambda^*$ which implies that $\Gamma(B) = \{\lambda^*\}$. Assume next that $\bar{b} \in \text{range}(A_B)$. We will show that $\Gamma(B)$ is open and hence an open interval. Indeed, let $\bar{\lambda} \in \Gamma(B)$. Then we have $A_B x_B = b + \overline{\lambda b}$ for some $x_B \in \mathbf{R}^{|B|}_{++}$. Also, since $\bar{b} \in \text{range}(A_B)$, we have $A_B u = \bar{b}$ for some $u \in \mathbf{R}^{|B|}$. Since $\bar{x}_B + \theta u > 0$ for all sufficiently small real number θ and $A_B(x_B + \theta u) = b + (\bar{\lambda} + \theta)\bar{b}$, it follows that $\bar{\lambda} + \theta \in \Gamma(B)$ for all sufficiently small θ. We have thus shown that any $\bar{\lambda} \in \Gamma(B)$ has a neighborhood contained in $\Gamma(B)$. Hence, $\Gamma(B)$ is an open interval. This completes the proof of (a) and (b). □

LEMMA 3.3. *The point $\lambda \in [\alpha, \beta]$ is a breakpoint of $\varphi(\lambda)$, that is, $\lambda = \lambda_i$ for some $i \in \{0, 1, \ldots, k\}$ if and only if $\bar{b} \notin \text{range}(A_B)$ where $B = B(\lambda)$.*

PROOF. Let $B = B(\lambda)$ and $N = N(\lambda)$. Clearly, $\lambda \in [\alpha, \beta]$ is a breakpoint of $\varphi(\lambda)$ if and only if $\varphi'_+(\lambda) > \varphi'_-(\lambda)$. By Proposition 2.4, the condition $\varphi'_+(\lambda) > \varphi'_-(\lambda)$ is equivalent to the condition that the linear function $y \to \bar{b}^T y$ is not constant on the optimal face $Y(\lambda) = \{y \mid A_B^T y = c_B;\ A_N^T y \leq c_N\}$. Since, by definition, the true equalities for the face $Y(\lambda)$ are the inequalities of the system $A_N^T y \leq c_N$, it follows from Proposition 2.1 that $y \to \bar{b}^T y$ is not constant on $Y(\lambda)$ if and only if $\bar{b} \notin \text{range}(A_B)$. Therefore, the result follows. □

We are now in a position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Statement (a) follows immediately from statements (a) and (b) of Lemma 3.1, statements (a) and (c) of Lemma 3.2, and from Lemma 3.3. We next show (b). We first show that $\bar{\Lambda}(B, N) \subseteq [\lambda_{i-1}, \lambda_i]$. By Proposition 2.3, it is enough to show that $\varphi(\lambda)$ is affine linear on $\bar{\Lambda}(B, N)$. Indeed, consider an arbitrary point y in $Y(\lambda^*)$. From (3.1), it follows that $y \in Y(\lambda)$ for all $\lambda \in \bar{\Lambda}(B, N)$. Hence, $\varphi(\lambda) = (b + \lambda \bar{b})^T y$ for all $\lambda \in \bar{\Lambda}(B, N)$ which proves our claim. By (a) of Lemma 3.1, (b) of Lemma 3.2, and Lemma 3.3, we know that $\Lambda(B, N)$ is an open interval, say (γ, δ). Since $\bar{\Lambda}(B, N) \subseteq [\lambda_{i-1}, \lambda_i]$, we have $(\gamma, \delta) \subseteq (\lambda_{i-1}, \lambda_i)$. We next show that $\gamma = \lambda_{i-1}$. Consider the optimality interval $\Lambda(\gamma) \equiv \Lambda(B(\gamma), N(\gamma))$. Clearly, $\Lambda(\gamma)$ is not an open interval since otherwise $\Lambda(B, N) \cap \Lambda(\gamma) \neq \emptyset$ which is a contradiction. Therefore, we must have $\gamma = \lambda_{i-1}$ since otherwise we would have $\gamma \in (\lambda_{i-1}, \lambda_i)$ and by the arguments above $\Lambda(\gamma)$ would be an open interval. The proof that $\delta = \lambda_i$ is analogous. Hence $\Lambda(B, N) = (\lambda_{i-1}, \lambda_i)$. From (b) of Lemma 3.1 and (c) of Lemma 3.2, it follows that $\bar{\Lambda}(B, N) = [\lambda_{i-1}, \lambda_i]$. □

Theorem 3.1 has a number of consequences which we now point out.

COROLLARY 3.1. *For $i \in \{1, \ldots, k\}$, all $Y(\lambda)$, $B(\lambda)$, and $N(\lambda)$ with $\lambda \in (\lambda_{i-1}, \lambda_i)$ do not change.*

PROOF. Immediate from Theorem 3.1. □

Henceforth, for all $\lambda \in (\lambda_{i-1}, \lambda_i)$, we denote the dual optimal faces $Y(\lambda)$ and the index sets $B(\lambda)$ and $N(\lambda)$ by Y_i, B_i, and N_i, respectively ($i \in \{1, \ldots, k\}$). Another consequence is the following.

COROLLARY 3.2. *For $i \in \{1, \ldots, k\}$, we have:*

(a) *$B(\lambda_{i-1})$ and $B(\lambda_i)$ is contained in B_i properly.*
(b) *$N(\lambda_{i-1})$ and $N(\lambda_i)$ contain N_i properly.*
(c) *$Y(\lambda_{i-1})$ and $Y(\lambda_i)$ contain Y_i properly.*
(d) *$Y(\lambda_i) \cap Y(\lambda_j) = \varnothing$ for all $i, j \in \{0, 1, \ldots, k\}$ with $|i - j| \geq 2$.*
(e) *$Y(\lambda_{i-1}) \cap Y(\lambda_i) = Y_i$.*
(f) *All dual optimal faces Y_i, $i \in \{1, \ldots, k\}$, are disjoint.*

PROOF. Clearly, statements (a), (b), and (c) are equivalent and they follow directly from Theorem 3.1 and the definition of the sets $\Lambda(B, N)$ and $\bar{\Lambda}(B, N)$. We next show (d). We may assume without loss of generality that $\lambda_i < \lambda_j$. First observe that

$$\varphi(\lambda_j) - \varphi(\lambda_i) = \sum_{l=i+1}^{j} g_l(\lambda_{l+1} - \lambda_l) > g_{i+1} \sum_{l=i+1}^{j} (\lambda_{l+1} - \lambda_l) = g_{i+1}(\lambda_j - \lambda_i).$$

To show (d), let $y \in Y(\lambda_i)$ be given. Then $\varphi(\lambda_i) = b^T y + \lambda_i \bar{b}^T y$ and, by Proposition 2.4, we also have $g_{i+1} \geq \bar{b}^T y$. Hence,

$$\varphi(\lambda_j) > \varphi(\lambda_i) + g_{i+1}(\lambda_j - \lambda_i) \geq (b^T y + \lambda_i \bar{b}^T y) + (\bar{b}^T y)(\lambda_j - \lambda_i) = b^T y + \lambda_j \bar{b}^T y,$$

which implies that $y \notin Y(\lambda_j)$. Thus, $Y(\lambda_i) \cap Y(\lambda_j) = \varnothing$ and (d) follows. We next show (e). Clearly, by (c), $Y(\lambda_{i-1}) \cap Y(\lambda_i) \supseteq Y_i$. To show that $Y(\lambda_{i-1}) \cap Y(\lambda_i) \subseteq Y_i$, let $y \in Y(\lambda_{i-1}) \cap Y(\lambda_i)$ be given. Then $\varphi(\lambda_{i-1}) = b^T y + \lambda_{i-1} \bar{b}^T y$ and $\varphi(\lambda_i) = b^T y + \lambda_i \bar{b}^T y$. Since $\varphi(\lambda)$ is affine linear on $[\lambda_{i-1}, \lambda_i]$, we can easily show that $\varphi(\lambda) = b^T y + \lambda \bar{b}^T y$ and hence that $y \in Y(\lambda)$ for any $\lambda \in (\lambda_{i-1}, \lambda_i)$. Since $Y_i = Y(\lambda)$ for $\lambda \in (\lambda_{i-1}, \lambda_i)$, we have $y \in Y_i$ and (e) follows. Statement (f) follows immediately from statements (c) and (d). □

It follows from (a) of Corollary 3.2 that $B(\lambda_{i-1}) \cup B(\lambda_i) \subseteq B_i$ for $i \in \{1, \ldots, k\}$. We leave the reader to verify that this inclusion may be proper for certain instances.

Yet another consequence of Theorem 3.1 is the following condition on the primal optimal faces $X(\lambda)$.

COROLLARY 3.3. *For all $i \in \{1, \ldots, k\}$ and $\gamma \in [0, 1]$, we have*

$$(1 - \gamma)X(\lambda_{i-1}) + \gamma X(\lambda_i) \subseteq X((1 - \gamma)\lambda_{i-1} + \gamma\lambda_i).$$

PROOF. Let $x^{i-1} \in X(\lambda_{i-1})$ and $x^i \in X(\lambda_i)$. Let $x(\gamma) = (1 - \gamma)x^{i-1} + \gamma x^i$ for $\gamma \in [0, 1]$. Clearly, $x(\gamma)$ is a feasible solution for problem (P_λ) with $\lambda = (1 - \gamma)\lambda_{i-1} +$

$\gamma\lambda_i$. Moreover, if we let $h(\gamma) = c^T x(\gamma)$, we have that $h(0) = \varphi(\lambda_{i-1})$, $h(1) = \varphi(\lambda_i)$ and, since both h and φ are affine functions on $[\lambda_{i-1}, \lambda_i]$, we must have $h(\gamma) = \varphi((1-\gamma)\lambda_{i-1} + \gamma\lambda_i)$ for all $\gamma \in [0, 1]$. Hence, $x(\gamma) \in X((1-\gamma)\lambda_{i-1} + \gamma\lambda_i)$ for all $\gamma \in [0, 1]$ and the result follows. □

Corollary 3.3 ensures that if we know optimal solutions corresponding to two consecutive breakpoints, then we can determine optimal solutions corresponding to any λ between these breakpoints.

Finally, we briefly comment on the similarities and differences between the results above and the existing theory of parametric RHS LP. Recall that the key to the analysis of the parametric RHS LP problem in the existing theory is $\bar{\Lambda}(B, N)$ where B is assumed to be a primal–dual basic vector with respect to (P_λ) for some $\lambda \in [\alpha, \beta]$ and $N = \{1, \ldots, n\} - B$. In particular, it has been shown (see, for example, Chapter 8 of [9]) that there exists a sequence of adjacent basic vectors $\tilde{B}_1, \ldots, \tilde{B}_l$ and points $\alpha = \tilde{\lambda}_0 \le \tilde{\lambda}_1 \le \cdots \le \tilde{\lambda}_{l-1} \le \tilde{\lambda}_l = \beta$ such that

$$\bar{\Lambda}(\tilde{B}_i, \tilde{N}_i) \cap \bar{\Lambda}(\tilde{B}_{i+1}, \tilde{N}_{i+1}) = \{\tilde{\lambda}_i\}, \qquad i = 1, \ldots, l-1,$$

$$\bigcup_{i=1}^{l} \bar{\Lambda}(\tilde{B}_i, \tilde{N}_i) = [\alpha, \beta],$$

where $\tilde{N}_i = \{1, \ldots, n\} - \tilde{B}_i$ for $i \in \{1, \ldots, l\}$. Indeed, under primal and dual nondegeneracy, the sequence of numbers $\tilde{\lambda}_i$ and the pair of index sets $(\tilde{B}_i, \tilde{N}_i)$ above are uniquely determined and they are precisely the sequence of numbers λ_i and the optimal partitions B_i, respectively, as described in the results above. The differences between the two theories stem from the presence of primal and/or dual degeneracies in the parametric RHS LP problem (P_λ), $\lambda \in \mathbf{R}$. In fact, while the partitions (B_i, N_i) and $(B(\lambda_i), N(\lambda_i))$ as described above are uniquely defined, the sequence of adjacent primal–dual basic vectors $\tilde{B}_i$, $i \in [1, \ldots, l]$, are no longer uniquely defined. Actually, it is possible to have a subsequence of intervals $\bar{\Lambda}(\tilde{B}_i)$ consisting of the same point (breakpoint) or a subsequence of these intervals covering an interval between two consecutive breakpoints. The significance of these differences will become more apparent in the discussion at the end of Section 4.

4. An Algorithm for the Parametric RHS LP Problem. In this section we present an algorithm for the parametric RHS LP problem. During the run of the algorithm, a sequence of LP problems is generated in such a way that the solution of one problem determines the next one to be solved. By solving this sequence of LP problems, we also obtain the solution to the given parametric RHS LP problem. As in the parametric RHS LP pivot algorithm, given a parameter $\lambda^* \in [\alpha, \beta]$ (see Proposition 2.3), the algorithm looks for the breakpoints of $\varphi(\lambda)$ first in one side of λ^* and then in the other side of λ^*.

By solving the parametric RHS LP problem, we mean:

(1) Find α and β satisfying the conditions of Proposition 2.3.
(2) Determine whether $\varphi([\alpha, \beta]) = \{-\infty\}$. If so, the solution to the parametric

RHS LP problem is completely specified and we stop. Otherwise, we have $\varphi([\alpha, \beta]) \subseteq \mathbf{R}$ and we continue to find the following additional information:

(a) the breakpoints $\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \beta$ of the function $\varphi(\lambda)$;
(b) the slopes g_i such that $g_i = \varphi'(\lambda)$ for $\lambda \in (\lambda_{i-1}, \lambda_i)$, $i \in \{1, \ldots, k\}$;
(c) the optimal partitions (B_i, N_i) such that $(B_i, N_i) = (B(\lambda), N(\lambda))$ for $\lambda \in (\lambda_{i-1}, \lambda_i)$, $i \in \{1, \ldots, k\}$;
(d) the optimal partitions $(B(\lambda_i), N(\lambda_i))$ for $i \in \{0, 1, \ldots, k\}$;
(e) $x^i \in X(\lambda_i)$ for $i \in \{0, 1, \ldots, k\}$;
(f) $y^i \in Y_i$ for $i \in \{1, \ldots, k\}$.

If we are interested in just finding the values $\varphi(\lambda)$ for all $\lambda \in [\alpha, \beta]$, all we need is (a) and (b) together with the value $\varphi(\lambda^*)$ for some $\lambda^* \in [\alpha, \beta]$. Points in the sets $X(\lambda)$ and $Y(\lambda)$ for all $\lambda \in [\alpha, \beta]$ can be obtained from information (e) and (f) by using the results of Corollaries 3.2 and 3.3.

In order to construct an algorithm that produces the required information, we present below some results which are consequences of the theory developed in Section 3.

As a consequence of Theorem 3.1 and the lemmas preceding it, we have the following observation. Given the optimal partition (B_i, N_i), $i \in \{1, \ldots, k\}$, we may compute the optimality interval $\Lambda(B_i, N_i) = (\lambda_{i-1}, \lambda_i)$ by solving the following LP problems:

$$(BR_i^{\min}) \qquad \lambda_{i-1} = \min\{\lambda \mid A_{B_i} x_{B_i} - \lambda \bar{b} = b;\ x_{B_i} \geq 0\},$$

$$(BR_i^{\max}) \qquad \lambda_i = \max\{\lambda \mid A_{B_i} x_{B_i} - \lambda \bar{b} = b;\ x_{B_i} \geq 0\}.$$

We may also compute the index sets $B(\lambda_{i-1})$ and $B(\lambda_i)$ by using the following corollary of Theorem 3.1 and Lemma 3.1.

COROLLARY 4.1. *The index set $B(\lambda_{i-1})$ (resp. $B(\lambda_i)$) is the set of all indices $j \in B_i$ such that $x_j \geq 0$ is a true inequality for the optimal face of problem $(BR_i^{\min})$ (resp. $(BR_i^{\max})$).*

PROOF. From (c) of Lemma 3.1 and the fact that $\lambda_{i-1}, \lambda_i \in \bar{\Lambda}(B_i, N_i)$, the optimal faces of problems $(BR_i^{\min})$ and $(BR_i^{\max})$ are respectively

$$\text{optimal face of } (BR_i^{\min}) = \{x_B \mid x \in X(\lambda_{i-1})\} \times \{\lambda_{i-1}\},$$

$$\text{optimal face of } (BR_i^{\max}) = \{x_B \mid x \in X(\lambda_i)\} \times \{\lambda_i\},$$

from which the result easily follows. □

Let λ_i, $i \in \{1, \ldots, k\}$, be a breakpoint of $\varphi(\lambda)$ and consider the two LP problems (see Proposition 2.4) as follows:

$$(SL_i^{\min}) \quad \min\{\bar{b}^T y \mid y \in Y(\lambda_i)\} = \min\{\bar{b}^T y \mid A_{B(\lambda_i)}^T y = c_{B(\lambda_i)};\ A_{N(\lambda_i)}^T y \leq c_{N(\lambda_i)}\},$$

$$(SL_i^{\max}) \quad \max\{\bar{b}^T y \mid y \in Y(\lambda_i)\} = \max\{\bar{b}^T y \mid A_{B(\lambda_i)}^T y = c_{B(\lambda_i)};\ A_{N(\lambda_i)}^T y \leq c_{N(\lambda_i)}\}.$$

We then have the following result.

THEOREM 4.1. *The optimal faces of problems* $(SL_i^{\min})$ *and* $(SL_i^{\max})$ *are equal to the optimal faces* Y_i *and* Y_{i+1}, *respectively.*

PROOF. Let $\hat{Y}_i$ denote the optimal face of $(SL_i^{\min})$. To show that $\hat{Y}_i = Y_i$, we only need to show that $\hat{Y}_i = Y(\lambda)$ for some $\lambda \in (\lambda_{i-1}, \lambda_i)$. First note that since $\varphi(\lambda)$ is affine linear on $[\lambda_{i-1}, \lambda_i]$, we have

$$\begin{aligned}\varphi(\lambda) &= \varphi(\lambda_i) + (\lambda - \lambda_i)\varphi'(\lambda)\\ &= \varphi(\lambda_i) + (\lambda - \lambda_i)g_i\end{aligned} \tag{4.1}$$

for any $\lambda \in (\lambda_{i-1}, \lambda_i)$. We first show that $\hat{Y}_i \subseteq Y(\lambda)$. Let $\hat{y} \in \hat{Y}_i$ be given. Then $\hat{y} \in Y(\lambda_i)$ and $\bar{b}^T\hat{y} = g_i$ by Proposition 2.4. Therefore,

$$\begin{aligned}(b + \lambda\bar{b})^T\hat{y} &= (b + \lambda_i\bar{b})^T\hat{y} + (\lambda - \lambda_i)\bar{b}^T\hat{y}\\ &= \varphi(\lambda_i) + (\lambda - \lambda_i)g_i = \varphi(\lambda),\end{aligned}$$

which implies that $\hat{y} \in Y(\lambda)$. We next show that $Y(\lambda) \subseteq \hat{Y}_i$. Let $\hat{y} \in Y(\lambda)$. Then $\hat{y} \in Y(\lambda_i)$ by Corollary 3.2 and $(b + \lambda\bar{b})^T\hat{y} = \varphi(\lambda)$. Therefore,

$$\begin{aligned}\bar{b}^T\hat{y} &= [(b + \lambda\bar{b})^T - (b + \lambda_i\bar{b})^T\hat{y}]/(\lambda - \lambda_i)\\ &= [\varphi(\lambda) - \varphi(\lambda_i)]/(\lambda - \lambda_i) = g_i.\end{aligned}$$

Since, by Proposition 2.4, g_i is the optimal value of $(SL_i^{\min})$, if follows that $\hat{y} \in \hat{Y}_i$. We have thus shown that $\hat{Y}_i = Y_i$. The proof that the optimal face of $(SL_i^{\max})$ is equal to Y_{i+1} is analogous. □

As a consequence of the previous result, we have the following corollary.

COROLLARY 4.2. *The set of true inequalities for the optimal face of problem* $(SL_i^{\min})$ *(resp.* $(SL_i^{\max})$*) is the set of all inequalities* $A_{\cdot j}^T y \le c_j$ *with* $j \in B_i$ *(resp.* $j \in B_{i+1}$*).*

PROOF. Immediate from Theorem 4.1. □

Using the results above, we are now in a position to construct an algorithm to solve the parametric RHS LP problem (P_λ), $\lambda \in \mathbf{R}$. The basis for this algorithm is an oracle, which for brevity we call Oracle A, for the following strong version of the LP problem.

ORACLE A. Given the following LP problem,

$$\min\{c^Tx \mid Ax \le b;\ Cx = d\}, \tag{P}$$

decide if (P) is infeasible, unbounded, or finite. If (P) is finite, find the set of true inequalities $A'x \le b'$ for the optimal face of (P) and a pair of primal and dual optimal solutions for (P).

Clearly, if we provide as input to Oracle A any of the problems (P_λ) with $\lambda \in [\alpha, \beta]$, as in Section 2, we obtain the optimal partition $(B(\lambda), N(\lambda))$. We discuss later how to implement such an oracle based on any LP solver.

We now describe the parametric RHS LP algorithm based on the results presented above. Assume that we have solved problem (P_λ) for some $\lambda \in [\alpha, \beta]$, say $\lambda = \lambda^*$. Determine if λ^* is a breakpoint by checking whether $\bar{b} \in \text{range}(A_{B(\lambda^*)})$. For simplicity, assume that λ^* is not a breakpoint and hence that $\lambda^* \in (\lambda_{i-1}, \lambda_i)$ for some $i \in \{1, \ldots, k\}$. We describe how to find the necessary information to the right of λ^*, that is, for all values $\lambda \in [\lambda^*, \beta]$. Clearly, λ_i can be determined as $\lambda_i = \max\{\lambda \mid A_B x_B - \lambda\bar{b} = b;\ x_B \geq 0\}$, where $(B, N) \equiv (B_i, N_i) = (B(\lambda^*), N(\lambda^*))$. Moreover, by Corollary 4.1, the set of true inequalities for the optimal face of this LP subproblem determines the optimal partition $(B(\lambda_i), N(\lambda_i))$. At the breakpoint λ_i, it follows from Proposition 2.4 that we can compute the slope g_{i+1} as $g_{i+1} = \max\{\bar{b}^T y \mid A_B^T y = c_B;\ A_N^T y \leq c_N\}$ where now $(B, N) \equiv (B(\lambda_i), N(\lambda_i))$. Moreover, by Corollary 4.2, the set of true inequalities for the optimal face of this LP subproblem determines the optimal partition (B_{i+1}, N_{i+1}). We are now in the same position as at the start and, by repeating the cycle described above, the breakpoints $\lambda_i, \ldots, \lambda_k$ and the slopes $g_{i+1}, \ldots, g_k$ will be determined consecutively. Note that the sequence of subproblems that are solved consecutively is (according to the notation adopted previously) $(BR_i^{\max})$, $(SL_i^{\max})$, $(BR_{i+1}^{\max})$, $(SL_{i+1}^{\max}), \ldots, (BR_k^{\max})$, and $(SL_k^{\max})$. The information to the left of λ^* can be found in a similar way by using a suitable sequence of LP subproblems in minimization form, more specifically the sequence $(BR_i^{\min})$, $(SL_{i-1}^{\min})$, $(BR_{i-1}^{\min})$, $(SL_{i-2}^{\min}), \ldots, (BR_1^{\min})$, and $(SL_0^{\min})$. The situation for the case in which λ^* is a breakpoint is very similar to the one above and therefore the details are omitted.

We next discuss how we can implement Oracle A. Toward this end, we first state a result presented in [3] which shows how to identify the set of true inequalities of a system of linear constraints by solving exactly one LP problem. Consider the following system of linear constraints:

$$Ax \leq b, \qquad Cx = d, \tag{4.2}$$

where $A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^m$, and $d \in \mathbf{R}^p$ and consider the following LP problem associated with (4.2):

$$\begin{aligned} &\text{maximize } e^T u \\ &\text{subject to} \quad Ax + u - b\alpha \leq 0, \\ &\qquad\qquad\qquad Cx - \alpha d = 0, \\ &\qquad\qquad\qquad 0 \leq u \leq e, \\ &\qquad\qquad\qquad \alpha \geq 1, \end{aligned} \tag{4.3}$$

where the variables are $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}$, and $e \in \mathbb{R}^m$ is the vector of all ones. Note that if problem (4.3) is feasible then it is finite. We can state the result as follows.

PROPOSITION 4.1 [3]. *The system* (4.2) *is feasible if and only if the LP problem* (4.3) *is feasible* (*and hence finite*). *In this case, any optimal solution* $(\bar{x}, \bar{u}, \bar{\alpha})$ *of* (4.3) *satisfies*

$$\bar{u}_i = \begin{cases} 1 & \text{if the } i\text{th inequality of } Ax \le b \text{ is a true inequality of (4.2)}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\bar{x}/\bar{\alpha}$ *is in the relative interior of the polyhedron defined by* (4.2).

Hence, an optimal solution of (4.3) completely determines the set of true inequalities of (4.2). For further discussion on how to identify the set of true inequalities of a linear constraint system see the brief note by Freund *et al.* [3]. Using the previous proposition, we can now discuss two approaches to implement Oracle A.

APPROACH 1. Solve the LP problem (P) and let θ^* denote the optimal value of (P). Next, apply the result of Proposition 4.1 to obtain the set of true inequalities of $Ax \le b$, $Cx = d$, and $c^Tx = \theta^*$. The resulting true inequalities are exactly the true inequalities for the optimal face of problem (P).

APPROACH 2. Combine both the primal and dual problems to obtain an equivalent feasiblity problem in the form of a linear constraint system and apply the result of Proposition 4.1 to determine the true inequalities of this system. Those inequalities of $Ax \le b$ that are true inequalities for the above system are exactly the true inequalities for the optimal face of problem (P).

As a consequence of the above discussion, we obtain the following complexity result.

THEOREM 4.2. *If there are k breakpoints in the piecewise linear function* $\varphi(\lambda)$ (*describing the optimal objective value as a function of the single parameter* λ *for the right-hand side vector*), *then the complexity of generating the entire function and the optimal faces is* $O(kZ)$ *where* Z *is the complexity of solving a single LP of the same dimension.*

Assume that the data of the parametric RHS LP problem is rational and let L denote the size of the problem, that is, the number of bits required to represent the data of the problem. It has been shown in [8] that there exists a class of parametric problems with a constraint matrix of dimension $n \times 2n$ $(n = 1, 2, \ldots)$ for which $O(L) = n$ and the number of breakpoints k of $\varphi(\lambda)$ is exponential in n. Hence, the parametric RHS LP problem clearly cannot be solved in a time bounded by a polynomial function of n and L. However, if k, the number of breakpoints of $\varphi(\lambda)$, is bounded by a polynomial function of L and n, then it follows from Theorem 4.2 and the existence of polynomial algorithms for solving LP problems that the parametric RHS LP problem (4.1) can be solved in polynomial time. The above result is clearly an improvement over the complexity of the based simplex algorithm for the parametric RHS LP problem. Roughly speaking, the sequence of basic vectors generated by the parametric RHS LP simplex algorithm can be viewed as solving alternately the LP problems $(BR_i^{\max})$ (or $(BR_i^{\min})$) and $(SL_i^{\max})$ (or $(SL_i^{\min})$). However, this algorithm may generate an exponential number of basic vectors to solve any one of these LP problems which is a property shared by some variants of the simplex algorithm. It is exactly under these circumstances that our parametric RHS LP algorithm may show substantial complexity improvement if Oracle A is implemented via a polynomial-time LP algorithm to solve the LP problems $(BR_i^{\max})$ (or $(BR_i^{\min})$) and $(SL_i^{\max})$ (or $(SL_i^{\min})$).

5. Remarks. In this section we provide some remarks as follows.

(1) Both of the approaches mentioned above to implement Oracle A have practical drawbacks. The first one involves solving two LP problems while the second one combines two LP problems into a larger one. We believe that a more promising approach is to solve the given LP problem by an interior-point method and be able to determine the set of true inequalities for the optimal face upon termination of the algorithm. This belief is justified by the fact that some interior-point methods are guaranteed to generate a sequence of interior points converging to (or accumulating at) some interior point in the relative interior of the optimal face of the LP problem. Interior-point path-following algorithms using short step size can be shown to satisfy this property. However, the assumption of using short step size is still very restrictive and most of the algorithms implemented in practice use long step size. In [2], an interior-point algorithm, known as the affine-scaling algorithm, which uses long step size, was shown to converge to the relative interior of the optimal face under primal nondegeneracy assumption.

(2) It should be noted that the knowledge of the optimal partition (B, N) of an LP provides extra information on the nature of the set of all optimal solutions. We believe that this extra information can be valuable for LP practitioners and should be a welcomed addition to the analysis of any parametric RHS LP problem.

(3) It is easy to see that under primal–dual nondegeneracy the following statements are true:

(a) $X(\lambda)$ is a singleton consisting of a vertex of the polyhedron $\{x \mid Ax = b + \lambda\bar{b}, x \geq 0\}$ for all $\lambda \in [\alpha, \beta]$.

(b) $Y(\lambda)$ is a singleton consisting of a vertex of the polyhedron $\{y \mid A^T y \leq c\}$ for any $\lambda \in (\lambda_{i-1}, \lambda_i)$ and $i \in \{1, \ldots, k\}$ and $Y(\lambda)$ is an edge of the polyhedron $\{y \mid A^T y \leq c\}$ for any $\lambda \in \{\lambda_0, \lambda_1, \ldots, \lambda_k\}$.

(c) B_i is the unique primal–dual feasible basic vector associated with the vertex $Y(\lambda)$ for any $\lambda \in (\lambda_{i-1}, \lambda_i)$ and $i \in \{1, \ldots, k\}$.

Thus, under primal and dual nondegeneracy, the LP problems $(BR_i^{\max})$ (or $(BR_i^{\min})$) and $(SL_i^{\max})$ (or $(SL_i^{\min})$) can be solved trivially as follows. Solving the LP problem $(BR_i^{\min})$ and $(BR_i^{\max})$ in this case means finding the optimality interval for the basic vector B_i which can be easily computed by using a ratio test procedure (see, for example, Chapter 8 of [9]) in $O(n)$ arithmetic operations. The LP problems $(SL_i^{\min})$ and $(SL_i^{\max})$ in this case involves minimizing a linear function over an edge of the polyhedron $\{y \mid A^T y \leq c\}$. Thus, this LP problem can be easily solved by means of a pivot step (in $O(n^2)$ arithmetic operations) which is exactly the step performed by the parametric RHS LP simplex algorithm.

Hence, under primal and dual nondegeneracy, both approaches execute the same steps to cover the interval $[\alpha, \beta]$.

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