An adaptive superfast inexact proximal augmented Lagrangian method for smooth nonconvex composite optimization problems

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Abstract

This work presents an adaptive superfast proximal augmented Lagrangian (AS-PAL) method for solving linearly-constrained smooth nonconvex composite optimization problems. Each iteration of AS-PAL inexactly solves a possibly nonconvex proximal augmented Lagrangian (AL) subproblem obtained by an aggressive/adaptive choice of prox stepsize with the aim of substantially improving its computational performance followed by a full Lagrangian multiplier update. A major advantage of AS-PAL compared to other AL methods is that it requires no knowledge of parameters (e.g., size of constraint matrix, objective function curvatures, etc) associated with the optimization problem, due to its adaptive nature not only in choosing the prox stepsize but also in using a crucial adaptive accelerated composite gradient variant to solve the proximal AL subproblems. The speed and efficiency of AS-PAL is demonstrated through extensive computational experiments showing that it can solve many instances more than ten times faster than other state-of-the-art penalty and AL methods, particularly when high accuracy is required.

1 Introduction

The main goal of this paper is to present the theoretical analysis and the excellent computational performance of an adaptive superfast proximal augmented Lagrangian method, referred to as AS-PAL, for solving the linearly-constrained smooth nonconvex composite optimization (SNCO) problem

$$\phi^* := \min\{\phi(z) := f(z) + h(z) : Az = b\}, \tag{1}$$

where $A : \Re^n \to \Re^l$ is a linear operator, $b \in \Re^l$, $h : \Re^n \to (-\infty, \infty]$ is a closed proper convex function which is M_h-Lipschitz continuous on its compact domain, and $f : \Re^n \to \Re$ is a real-valued differentiable nonconvex function which is m_f-weakly convex and whose gradient is L_f–Lipschitz continuous. AS-PAL is essentially an adaptive version of the IAIPAL method and the NL-IAIPAL method studied in [15, 16], but, in contrast to these methods, it does not require knowledge of the above parameters m_f, L_f, and M_h.

An iteration of AS-PAL has a similar pattern to the ones of the methods in [15, 16] and is also based on the augmented Lagrangian (AL) function $\mathcal{L}_c(z; p)$ defined as

$$\mathcal{L}_c(z; p) := f(z) + h(z) + \langle p, Az - b\rangle + \frac{c}{2}\|Az - b\|^2. \tag{2}$$

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More specifically, its rough description is as follows: given $(z_{k-1}, p_{k-1}) \in \mathcal{H} \times \Re^l$ and a pair of positive scalars (λ_k, c_k), it computes z_k as a suitable approximate solution of the possibly nonconvex proximal subproblem

$$\min_u \left\{ \lambda_k \mathcal{L}_{c_k}(u; p_{k-1}) + \frac{1}{2}\|u - z_{k-1}\|^2 \right\}, \tag{3}$$

and p_k according to the full Lagrange multiplier update

$$p_k = p_{k-1} + c_k(Az_k - b). \tag{4}$$

Based on the fact that (3) is strongly convex whenever the prox stepsize λ_k is chosen in $(0, 1/m_f)$, the methods of [15, 16] set $\lambda_k = 0.5/m_f$ for every k and solve each strongly-convex subproblem using an accelerated composite gradient (ACG) method (see [12, 27]).

Our contributions: Since it is empirically observed that the larger λ_k is, the faster the procedure outlined above in (3)-(4) approaches a desired approximate solution of (1), AS-PAL adaptively chooses the prox stepsize λ_k to be a scalar which is usually much larger than $0.5/m_f$. As (3) may become nonconvex with such a choice of λ_k, a standard ACG method applied to (3) may fail to obtain a desirable approximate solution of (3). To remedy this situation, AS-PAL uses a new adaptive ACG method for solving (3) which accounts for the fact that (3) may be nonconvex and the Lipschitz constant of the objective function of (3) may be unknown. Thus, in contrast to the methods of [15, 16], AS-PAL has the interesting feature of requiring no knowledge of the parameters m_f, L_f and M_h underlying (3) in view of its ability to adaptively generate the prox stepsize λ_k and the estimate of the Lipschitz constant of the objective function of (3) within the adaptive ACG method. Moreover, as was shown for the method of [15], under the assumption that a Slater point exists, it is also shown that, for any given tolerance pair $(\hat{\rho}, \hat{\eta}) \in \Re^2_{++}$, AS-PAL finds a $(\hat{\rho}, \hat{\eta})$-approximate stationary solution of (3), i.e., a triple (z, p, w) satisfying

$$w \in \nabla f(z) + \partial h(z) + A^* p, \quad \|w\| \le \hat{\rho}, \quad \|Az - b\| \le \hat{\eta}, \tag{5}$$

in at most $\mathcal{O}(\hat{\eta}^{-1/2}\hat{\rho}^{-2} + \hat{\rho}^{-3})$ iterations (up to logarithmic terms). Finally, a major advantage of AS-PAL is that it substantially improves the computational performance of the methods in [15, 16], whose performance was already substantially better than other existing methods for solving (1). Our extensive computational results of section 4 show that AS-PAL can efficiently compute highly accurate solutions for all problems tested, while the other methods can fail to do so in many of these problems. AS-PAL can often find such solutions in just a few seconds or minutes while all the other methods may take several hours to do so.

Literature review. We only focus on relatively recent papers dealing with the iteration complexity of augmented Lagrangian (AL) type methods. In the convex setting, AL-based methods have been widely studied for example in [1, 2, 19, 20, 24, 25, 28, 31, 34].

We now discuss AL type methods in the nonconvex setting of (1). Various proximal AL methods for solving both linearly and nonlinearly constrained SNCO problems have been studied in [6, 15, 16, 17, 26, 33, 36, 37, 38]. More specifically, [6, 17, 26] present proximal AL methods based on a perturbed augmented Lagrangian function and an under-relaxed multiplier update. Papers [15, 16] present an accelerated proximal AL method based off the classical augmented Lagrangian function and a full multiplier update. The method in [33] is an AL-based method which reverses the direction of the multiplier update. Papers [36, 37, 38] study AL type variants based on the Moreau envelope. Finally, non-proximal AL methods for solving SNCO problems are studied in [21, 32].

We now discuss papers that are tangentially related to this work. Penalty methods for SNCO problems have been studied in [13, 14, 18, 23]. It is worth mentioning that AS-PAL extends the methods in [15, 16] by allowing for an adaptive prox stepsize, similar to the way the method of [14] extends the one in [13]. Finally, paper [9] studies a penalty-ADMM method that solves an equivalent reformulation of (1) while the paper [22] presents an inexact proximal point method applied to the function defined as $\phi(z)$ if z is feasible and $+\infty$ otherwise.

Organization of the paper. The paper is laid out as follows. Subsection 1.1 presents basic definitions and notation used throughout the paper. Section 2 contains two subsections. The first describes the problem of interest and the assumptions made on it. The second formally states the AS-PAL method and its main complexity result. Section 3 is dedicated to proving the main complexity result. Section 4 presents extensive computational experiments which demonstrate the efficiency of AS-PAL. The Appendix contains two subsections. Appendix A presents the ADAP-FISTA algorithm which is used to solve possibly nonconvex unconstrained subproblems while Appendix B presents technical results which are used to prove that the sequence of the Lagrange multipliers generated by AS-PAL is bounded.

1.1 Basic Definitions and Notations

This subsection presents notation and basic definitions used in this paper.

Let $\Re_+$ and $\Re_{++}$ denote the set of nonnegative and positive real numbers, respectively. We denote by $\Re^n$ an n-dimensional inner product space with inner product and associated norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. We use $\Re^{l\times n}$ to denote the set of all $l\times n$ matrices and $\mathbb{S}^+_n$ to denote the set of positive semidefinite matrices in $\Re^{n\times n}$. The smallest positive singular value of a nonzero linear operator $Q:\Re^n\to\Re^l$ is denoted by ν^+_Q. For a given closed convex set $Z\subset\Re^n$, its boundary is denoted by ∂Z and the distance of a point $z\in\Re^n$ to Z is denoted by $\mathrm{dist}(z,Z)$. The indicator function of Z, denoted by δ_Z, is defined by $\delta_Z(z)=0$ if $z\in Z$, and $\delta_Z(z)=+\infty$ otherwise. For any $t>0$ and $b\geq 0$, we let $\log^+_b(t):=\max\{\log t, b\}$, and we define $\mathcal{O}_1(\cdot)=\mathcal{O}(1+\cdot)$.

The domain of a function $h:\Re^n\to(-\infty,\infty]$ is the set $\mathrm{dom}\, h:=\{x\in\Re^n : h(x)<+\infty\}$. Moreover, h is said to be proper if $\mathrm{dom}\, h\neq\emptyset$. The set of all lower semi-continuous proper convex functions defined in $\Re^n$ is denoted by $\overline{\mathrm{Conv}}\,\Re^n$. The ε-subdifferential of a proper function $h:\Re^n\to(-\infty,\infty]$ is defined by

$$\partial_\varepsilon h(z):=\{u\in\Re^n : h(z')\geq h(z)+\langle u, z'-z\rangle-\varepsilon,\quad\forall z'\in\Re^n\}\tag{6}$$

for every $z\in\Re^n$. The classical subdifferential, denoted by $\partial h(\cdot)$, corresponds to $\partial_0 h(\cdot)$. Recall that, for a given $\varepsilon\geq 0$, the ε-normal cone of a closed convex set C at $z\in C$, denoted by $N^\varepsilon_C(z)$, is

$$N^\varepsilon_C(z):=\{\xi\in\Re^n : \langle\xi, u-z\rangle\leq\varepsilon,\quad\forall u\in C\}.$$

The normal cone of a closed convex set C at $z\in C$ is denoted by $N_C(z)=N^0_C(z)$. If ψ is a real-valued function which is differentiable at $\bar z\in\Re^n$, then its affine approximation $\ell_\psi(\cdot,\bar z)$ at $\bar z$ is given by

$$\ell_\psi(z;\bar z):=\psi(\bar z)+\langle\nabla\psi(\bar z), z-\bar z\rangle\quad\forall z\in\Re^n.\tag{7}$$

2 The AS-PAL method

This section consists of two subsections. The first one precisely describes the problem of interest and its assumptions. The second one motivates and states the AS-PAL method and presents its main complexity result.

2.1 Problem of Interest

This subsection presents the main problem of interest and discusses the assumptions underlying it.

Consider problem (1) where $A : \Re^n \to \Re^l$, $b \in \Re^l$ and functions $f, h : \Re^n \to (-\infty, \infty]$ satisfy the following assumptions:

(A1) $h \in \overline{\text{Conv}}(\Re^n)$ is M_h-Lipschitz continuous on $\mathcal{H} := \text{dom}\, h$ and the diameter

$$D_h := \sup\{\|z - z'\| : z, z' \in \mathcal{H}\}$$

of $\mathcal{H}$ is finite;

(A2) A is a nonzero linear operator and there exists $\bar{z} \in \text{int}\,(\mathcal{H})$ such that $A\bar{z} = b$;

(A3) f is nonconvex and differentiable on $\Re^n$, and there exists $L_f \geq m_f > 0$ such that for all $z, z' \in \Re^n$,

$$\|\nabla f(z') - \nabla f(z)\| \leq L_f \|z' - z\|, \tag{8}$$

$$f(z') - \ell_f(z'; z) \geq -\frac{m_f}{2}\|z' - z\|^2. \tag{9}$$

2.2 The AS-PAL method

This subsection motivates and states the AS-PAL method and presents its main complexity result.

Before presenting the method, we give a short but precise outline of its key steps as well as a description of how its iterates are generated. Recall from the introduction that the AS-PAL method, whose goal is to find a $(\hat{\rho}, \hat{\eta})$-approximate stationary solution as in (5), is an iterative method which, at its k-th step, computes its next iterate (z_k, p_k) according to (3) and (4). We are now ready to provide a complete description of the AS-PAL method.

AS-PAL Method

Input: functions (f, h), scalars $\sigma \in (0, 1/2)$, $\chi \in (0, 1)$, and $\beta > 1$, an initial $\bar{\lambda} > 0$, an initial point $z_0 \in \mathcal{H}$, $p_0 = 0$, a penalty parameter $c_1 > 0$, and a tolerance pair $(\hat{\rho}, \hat{\eta}) \in \Re^2_{++}$.
Output: a triple (z, p, w) satisfying (5).

0. set $\hat{k} = 1$, $k = 1$, and

$$\lambda = \bar{\lambda}, \quad C_\sigma = \frac{2(1-\sigma)^2}{1 - 2\sigma}; \tag{10}$$

1. let $M_0^k \in [1, \bar{\lambda}(L_f + c_k\|A\|^2) + 1]$ and call the ADAP-FISTA method described in Appendix A with inputs

$$x_0 = z_{k-1}, \quad (\mu, L_0, \chi, \beta, \sigma) = (1/2, M_0^k, \chi, \beta, \sigma), \tag{11}$$

$$\psi_s = \lambda[\mathcal{L}_{c_k}(\cdot, p_{k-1}) - h] + \frac{1}{2}\|\cdot - z_{k-1}\|^2, \quad \psi_n = \lambda h; \tag{12}$$

2. if ADAP-FISTA fails or its output (z, u) (if it succeeds) does not satisfy the inequality

$$\lambda \mathcal{L}_{c_k}(z_{k-1}, p_{k-1}) - \left[\lambda \mathcal{L}_{c_k}(z, p_{k-1}) + \frac{1}{2}\|z - z_{k-1}\|^2\right] \geq \langle u, z_{k-1} - z \rangle, \tag{13}$$

then set $\lambda = \lambda/2$ and go to step 1; else, set $(z_k, u_k) = (z, u)$, $\lambda_k = \lambda$, and

$$w_k := \frac{u_k + z_{k-1} - z_k}{\lambda_k}, \tag{14}$$

$$p_k := p_{k-1} + c_k(Az_k - b), \tag{15}$$

and go to step 3;

3. if $\|w_k\| \le \hat{\rho}$ and $\|Az_k - b\| \le \hat{\eta}$, then stop with success and output $(z, p, w) = (z_k, p_k, w_k)$; else, go to step 4;

4. if $k \ge \hat{k} + 1$ and

$$\Delta_k := \frac{1}{\sum_{i=\hat{k}+1}^{k} \lambda_i} \left[\mathcal{L}_{c_k}(z_{\hat{k}}, p_{\hat{k}-1}) - \mathcal{L}_{c_k}(z_k, p_k) - \frac{\|p_k\|^2}{2c_k} \right] \le \max\left\{ \frac{\sum_{i=\hat{k}+1}^{k} \lambda_i \|w_i\|^2}{2C_\sigma \sum_{i=\hat{k}+1}^{k} \lambda_i}, \frac{\hat{\rho}^2}{2C_\sigma} \right\}, \tag{16}$$

then set $c_{k+1} = 2c_k$ and $\hat{k} = k + 1$; otherwise, set $c_{k+1} = c_k$;

5. set $k \leftarrow k + 1$ and go to step 1.

AS-PAL makes two types of iterations, namely, the outer iterations indexed by k and the ACG iterations performed during its calls to the ADAP-FISTA method in step 1.

We now make some remarks about AS-PAL. First, it follows from Proposition A.1 (see Appendix A) that the total number of resolvent evaluations [1] made by ADAP-FISTA is on the same order of magnitude as its total number of ACG iterations. Second, noting that the sum of the functions ψ_s and ψ_n in (12) is equal to the objective function of (3), it follows from Proposition A.1 in Appendix A that the pair (z_k, u_k) in step 2 of AS-PAL is an approximate solution of (3) in the sense of (51). Third, it will be shown in Proposition 3.1(b) below that the triple (z_k, p_k, w_k) computed in step 2 satisfies the inclusion in (5) for every $k \ge 1$. As a consequence, if AS-PAL terminates in step 3, then the triple (z, p, w) output in this step is a $(\hat{\rho}, \hat{\eta})$-approximate solution of (1). Finally, step 4 provides a test, namely, inequality (16), to determine when to increase the penalty parameter c_k.

Define the l-th cycle $\mathcal{C}_l$ as the l-th set of consecutive indices $k \ge 1$ for which c_k remains constant, i.e.,

$$\mathcal{C}_l := \{k \ge 1 : c_k = \tilde{c}_l := 2^{l-1} c_1\} \quad \forall l \ge 1. \tag{17}$$

For every $l \ge 1$, let k_l denote the smallest index in $\mathcal{C}_l$. Hence,

$$\mathcal{C}_l = \{k_l, \ldots, k_{l+1} - 1\} \quad \forall l \ge 1. \tag{18}$$

Clearly, the different values of $\hat{k}$ that arise in step 4 are exactly the indices in $\{k_l : l \ge 1\}$. Moreover, in view of the test performed in step 4, we have that $k_{l+1} - k_l \ge 2$ for every $l \ge 1$, or equivalently, every cycle contains at least two indices. While generating the indices in the l-th cycle, if an index $k \ge k_l + 2$ satisfying (16) is found, k becomes the last index $k_{l+1} - 1$ in the l-th cycle and the $(l+1)$-th cycle is started at iteration k_{l+1} with the penalty parameter set to $\tilde{c}_{l+1} = 2\tilde{c}_l$, where $\tilde{c}_l$ is as in (17).

In the remaining part of this section, we state the main complexity result for AS-PAL, whose proof is the main focus of Section 3. Before stating the main result, we first introduce the following quantities:

$$\phi_* := \inf_{z \in \Re^n} \phi(z), \quad \bar{d} := \operatorname{dist}(\bar{z}, \partial\mathcal{H}), \quad \underline{\lambda} := \min\{\overline{\lambda}, 1/(4m_f)\} \tag{19}$$

[1] A resolvent evaluation of h is an evaluation of $(I + \gamma\partial h)^{-1}(\cdot)$ for some $\gamma > 0$.

$$\nabla_f := \sup_{z\in\mathcal{H}} |\nabla f(z)|, \quad \kappa_p := \frac{2D_h(M_h + \nabla_f + \underline{\lambda}^{-1}(1+\sigma)D_h)}{\bar{d}\nu_A^+}, \quad \hat{c}(\hat{\rho},\hat{\eta}) := \frac{18C_\sigma\kappa_p^2}{\underline{\lambda}\hat{\rho}^2} + \frac{2\kappa_p}{\hat{\eta}} \tag{20}$$

$$S := \sup_{z\in\mathcal{H}} |\phi(z)|, \quad \kappa_d := S + \frac{4\kappa_p^2}{c_1} - \phi_*, \tag{21}$$

where $\overline{\lambda}$, c_1, and σ are input parameters for AS-PAL, (m_f, L_f) are as in (A3), $\bar{z}$ is as in (A2), M_h is as in (A1), D_h is as in (A1), ν_A^+ is as in Subsection 1.1, and C_σ is as in (10). Note that assumptions (A1) and (A3) imply that S and ∇_f are finite.

The following result describes the ACG iteration/resolvent evaluation complexity for AS-PAL.

Theorem 2.1. *Let a tolerance pair $(\hat{\rho},\hat{\eta}) \in \Re^2_{++}$ be given and assume that $c_1 \le 4\hat{c}(\hat{\rho},\hat{\eta})$ and $\overline{\lambda}$ is such that $\overline{\lambda} = \Omega(m_f^{-1})$ and $\log_0^+(m_f\overline{\lambda}) \le \mathcal{O}(1 + \kappa_d/(\underline{\lambda}\hat{\rho}^2))$, where c_1 and $\overline{\lambda}$ are the initial penalty parameter and prox stepsize of AS-PAL, respectively, m_f is as in (A3), $\hat{c}(\hat{\rho},\hat{\eta})$ is as in (20), and κ_d is as in (21). Then, AS-PAL outputs a $(\hat{\rho},\hat{\eta})$-approximate stationary solution of (1) in*

$$\mathcal{O}\left(\left[1 + \frac{m_f\kappa_d}{\hat{\rho}^2}\right]\sqrt{\mathcal{M}(\hat{c})}\left[\log\left(\mathcal{M}(\hat{c}) + \frac{\hat{c}}{c_1}\right)\right]^2\right) \tag{22}$$

ACG iterations/resolvent evaluations, where $\hat{c} := \hat{c}(\hat{\rho},\hat{\eta})$ and

$$\mathcal{M}(c) := \overline{\lambda}(L_f + c\|A\|^2) + 1 \quad \forall c \in \Re. \tag{23}$$

It follows from the definitions of $\hat{c}(\cdot,\cdot)$ and $\mathcal{M}(\cdot)$ in (20) and (23), respectively, that the iteration complexity bound (22) in terms of the tolerance pair $(\hat{\rho},\hat{\eta})$, up to a logarithmic term, is

$$\mathcal{O}\left(\frac{1}{\sqrt{\hat{\eta}}\cdot\hat{\rho}^2} + \frac{1}{\hat{\rho}^3}\right).$$

3 Proof of Theorem 2.1

The result below describes properties of the loop consisting of steps 1 and 2 of AS-PAL.

Proposition 3.1. *Let $k \in \mathcal{C}_l$ for some $l \ge 1$ be given. Then, the following statements hold:*

(a) every ACG call in step 1 of the k-th iteration of AS-PAL performs

$$\mathcal{O}_1\left(\sqrt{\mathcal{M}(\tilde{c}_l)}\,\log\mathcal{M}(\tilde{c}_l)\right) \tag{24}$$

ACG iterations/resolvent evaluations;

(b) during the k-th iteration of AS-PAL, the loop consisting of steps 1 and 2 eventually ends with a quintuple $(z_k, u_k, w_k, p_k, \lambda_k)$ satisfying

$$\|u_k\| \le \sigma\min\left\{\|z_k - z_{k-1}\|, \frac{\|\lambda_k w_k\|}{1-\sigma}\right\}; \tag{25}$$

$$\lambda_k\mathcal{L}_{c_k}(z_{k-1},p_{k-1}) - \left[\lambda_k\mathcal{L}_{c_k}(z_k,p_{k-1}) + \frac{1}{2}\|z_k - z_{k-1}\|^2\right] \ge \langle u_k, z_{k-1} - z_k\rangle; \tag{26}$$

$$w_k \in \nabla f(z_k) + \partial h(z_k) + A^*p_k, \quad \|\lambda_k w_k\| \le (1+\sigma)\|z_k - z_{k-1}\|; \tag{27}$$

$$\overline{\lambda} \ge \lambda_k \ge \underline{\lambda}, \tag{28}$$

where $\overline{\lambda}$ is the initial prox stepsize and $\underline{\lambda}$ is as in (19); moreover, every prox stepsize λ generated in the loop consisting of steps 1 and 2 of AS-PAL is in $[\underline{\lambda},\overline{\lambda}]$.

Proof. (a) Using the definition of $\mathcal{L}_{c_k}(\cdot; p_{k-1})$ in (2) and assumption (8), we easily see that its smooth part, namely, $\mathcal{L}_{c_k}(\cdot; p_{k-1}) - h(\cdot)$, has $(L_f + c_k\|A\|^2)$-Lipschitz continuous gradient everywhere on $\Re^n$. This observation together with the facts that $\lambda \le \overline{\lambda}$, $c_k = \tilde{c}_l$, and the definition of $\mathcal{M}(\cdot)$ in (23), then imply that the function ψ_s in (12) has $\mathcal{M}(\tilde{c}_l)$-Lipschitz continuous gradient. Noting that M_0^k in step 1 is chosen so that $M_0^k \le \mathcal{M}(c_k) = \mathcal{M}(\tilde{c}_l)$ and that each call to ADAP-FISTA in step 1 is made with $(\mu, L_0) = (1/2, M_0^k)$, we then conclude that (a) follows directly from Proposition A.1(a) with $\bar{L} = \mathcal{M}(\tilde{c}_l)$ and $\mu = 1/2$.

(b) We first claim that if the loop consisting of steps 1 and 2 of the k-iteration of AS-PAL stops, then (25), (26), and (27) hold. Indeed, assume that the loop consisting of steps 1 and 2 of the k-th iteration of AS-PAL stops. It then follows that ADAP-FISTA with inputs given by (11) and (12) stops successfully and $(z, u, \lambda) = (z_k, u_k, \lambda_k)$ satisfies (13). These two conclusions, identities (12) and (14), and Proposition A.1(b) with (ψ_s, ψ_n) as in (12), $x_0 = z_{k-1}$, and $(y, u) = (z_k, u_k)$ then imply that (26), the first inequality in (25), and the inclusion in (27) hold. Now, using the definition of w_k in (14), the triangle inequality, and the first inequality in (25), we have:

$$\frac{1}{\sigma}\|u_k\| - \|u_k\| \overset{(25)}{\le} \|z_k - z_{k-1}\| - \|u_k\| \le \|u_k + z_{k-1} - z_k\| \overset{(14)}{=} \|\lambda_k w_k\| \overset{(25)}{\le} (1+\sigma)\|z_k - z_{k-1}\|, \quad (29)$$

from which the second inequality in (25) and the inequality in (27) follow.

We now claim that if step 1 is performed with a prox stepsize $\lambda \le 1/(2m_f)$ in the k-th iteration, then for every $j > k$, we have that $\lambda_{j-1} = \lambda$ and the j-th iteration performs step 1 only once. To show the claim, assume that $\lambda \le 1/(2m_f)$. Using this assumption, the definition of $\mathcal{L}_c$ in (2), and the assumption (9) that f is m_f-weakly convex, we see that the function ψ_s in (12) is strongly convex with modulus $1 - \lambda m_f \ge 1/2$. Since each ACG call is performed in step 1 of AS-PAL with $\mu = 1/2$, it follows immediately from Proposition A.2 with (ψ_s, ψ_n) as in (12) that ADAP-FISTA terminates successfully and outputs a pair (z, u) satisfying $u \in \partial(\psi_s + \psi_n)(z)$. This inclusion, the definition of (ψ_s, ψ_n), and the definition of subdifferential in (6), then imply that (13) holds. Hence, in view of the termination criteria of step 2 of AS-PAL, it follows that $\lambda_k = \lambda$. It is then easy to see, by the way λ is updated in step 2 of AS-PAL, that λ is not halved in the $(k+1)$-th iteration or any subsequent iteration, hence proving the claim.

It is now straightforward to see that the above two claims, the fact that the initial value of the prox stepsize is equal to $\overline{\lambda}$, and the way λ_k is updated in AS-PAL, imply that the lemma holds. □

The subsequent technical result characterizes the change in the augmented Lagrangian function between consecutive iterations of the AS-PAL method.

Lemma 3.2. *For every $k \ge 1$, we have:*

$$\mathcal{L}_{c_k}(z_k, p_k) - \mathcal{L}_{c_k}(z_k, p_{k-1}) = \frac{1}{c_k}\|p_k - p_{k-1}\|^2, \quad (30)$$

and

$$\frac{\lambda_k}{C_\sigma}\|w_k\|^2 \le \mathcal{L}_{c_k}(z_{k-1}, p_{k-1}) - \mathcal{L}_{c_k}(z_k, p_k) + \frac{1}{c_k}\|p_k - p_{k-1}\|^2 \quad (31)$$

where C_σ is as in (10).

Proof. Identity (30) follows immediately from the definition of the Lagrangian in (2) and relation (15). Now, using relation (26), the second inequality in (25), and the definitions of C_σ and w_k in (10) and (14), respectively, we conclude that:

$$\lambda_k \mathcal{L}_{c_k}(z_{k-1}, p_{k-1}) - \lambda_k \mathcal{L}_{c_k}(z_k, p_{k-1}) \overset{(26)}{\ge} \frac{1}{2}\|z_k - z_{k-1}\|^2 + \langle u_k, z_{k-1} - z_k\rangle$$

$$= \frac{1}{2}\|z_{k-1} - z_k + u_k\|^2 - \frac{1}{2}\|u_k\|^2 \overset{(14)}{=} \frac{1}{2}\|\lambda_k w_k\|^2 - \frac{1}{2}\|u_k\|^2$$

$$\overset{(25)}{\geq} \frac{1}{2}\|\lambda_k w_k\|^2 - \frac{\sigma^2}{2(1-\sigma)^2}\|\lambda_k w_k\|^2 = \frac{1-2\sigma}{2(1-\sigma)^2}\|\lambda_k w_k\|^2 \overset{(10)}{=} \frac{\|\lambda_k w_k\|^2}{C_\sigma}. \tag{32}$$

Inequality (31) now follows by dividing (32) by λ_k and combining the resulting inequality with (30). □

The result below, which establishes boundedness of the sequence of Lagrange multipliers, makes use of a technical result in the Appendix, namely Lemma B.3.

Proposition 3.3. *The sequence $\{p_k\}$ generated by AS-PAL satisfies*

$$\|p_k\| \leq \kappa_p, \quad \forall k \geq 0, \tag{33}$$

where κ_p is defined in (20).

Proof. Using the inequality in (27), the triangle inequality, the second inequality in (28), and the definitions of D_h and ∇_f in (A1) and (20), respectively, we conclude that

$$\|w_k - \nabla f(z_k)\| \overset{(27)}{\leq} \frac{1}{\lambda_k}(1+\sigma)\|z_k - z_{k-1}\| + \nabla_f \overset{(28)}{\leq} \frac{D_h(1+\sigma)}{\underline{\lambda}} + \nabla_f. \tag{34}$$

Now, using the inclusion in (27), the relation in (34), Lemma B.3(b) with $(z, q, r) = (z_k, p_k, w_k - \nabla f(z_k))$ and $q^- = p_{k-1}$, and the definition of κ_p in (20), we conclude that for every $k \geq 1$:

$$\|p_k\| \overset{(87)}{\leq} \max\left\{\|p_{k-1}\|, \frac{2D_h(M_h + \|w_k - \nabla f(z_k)\|)}{\bar{d}\nu_A^+}\right\} \overset{(34)}{\leq} \max\left\{\|p_{k-1}\|, \kappa_p\right\}. \tag{35}$$

Now, the conclusion of the proposition follows from the above relation, the fact that $p_0 = 0$, and a simple induction argument. □

Recall that the l-th cycle $\mathcal{C}_l$ of AS-PAL is defined in (17). The following result shows that the sequence $\{\|\|w_k\|\}_{k\in\mathcal{C}_l}$ is bounded and can be controlled by $\{\Delta_k\}_{k\in\mathcal{C}_l}$ plus a term which is of $\mathcal{O}(1/\tilde{c}_l)$.

Lemma 3.4. *Consider the sequences $\{(z_k, p_k, w_k)\}_{k\in\mathcal{C}_l}$ and $\{\Delta_k\}$ generated by AS-PAL. Then, for every $k \in \mathcal{C}_l$ such that $k \geq \hat{k}+1$, we have:*

$$\frac{\sum_{i=\hat{k}+1}^{k} \lambda_i \|w_i\|^2}{\sum_{i=\hat{k}+1}^{k} \lambda_i} \leq C_\sigma \left(\Delta_k + \frac{9\kappa_p^2}{\underline{\lambda}\tilde{c}_l}\right) \tag{36}$$

where C_σ, $\underline{\lambda}$, and κ_p are as in (10), (19), and (20), respectively and $\hat{k}$ is the first index in $\mathcal{C}_l$.

Proof. We have by relation (33) and the bound $\|p_j - p_{j-1}\|^2 \leq 2\|p_j\|^2 + 2\|p_{j-1}\|^2 \leq 4\kappa_p^2$, that it follows that for any $k \in \mathcal{C}_l$,

$$\frac{\|p_k\|^2}{2} + \sum_{i=\hat{k}}^{k} \|p_i - p_{i-1}\|^2 \leq \frac{\kappa_p^2}{2} + 4(k - \hat{k} + 1)\kappa_p^2 = \frac{(1 + 8(k - \hat{k} + 1))\kappa_p^2}{2} \leq 9(k - \hat{k})\kappa_p^2. \tag{37}$$

Hence, relations (30), (31), and (37) and the fact that $c_k = \tilde{c}_l$ for every $k \in \mathcal{C}_l$, imply that for any $k \in \mathcal{C}_l$ such that $k \geq \hat{k}+1$,

$$\frac{1-2\sigma}{2(1-\sigma)^2} \sum_{i=\hat{k}+1}^{k} \lambda_i \|w_i\|^2 \overset{(31)}{\leq} \sum_{i=\hat{k}+1}^{k} \left[\mathcal{L}_{c_i}(z_{i-1}, p_{i-1}) - \mathcal{L}_{c_i}(z_i, p_i) + \frac{1}{c_i}\|p_i - p_{i-1}\|^2 \right]$$

$$\overset{j \in \mathcal{C}_l}{=} \sum_{i=\hat{k}+1}^{k} \left[\mathcal{L}_{\tilde{c}_l}(z_{i-1}, p_{i-1}) - \mathcal{L}_{\tilde{c}_l}(z_i, p_i) + \frac{1}{\tilde{c}_l} \|p_i - p_{i-1}\|^2 \right]$$

$$= \mathcal{L}_{\tilde{c}_l}(z_{\hat{k}}, p_{\hat{k}}) - \mathcal{L}_{\tilde{c}_l}(z_k, p_k) + \frac{1}{\tilde{c}_l} \sum_{i=\hat{k}+1}^{k} \|p_i - p_{i-1}\|^2$$

$$\overset{(30)}{=} \mathcal{L}_{\tilde{c}_l}(z_{\hat{k}}, p_{\hat{k}-1}) - \mathcal{L}_{\tilde{c}_l}(z_k, p_k) + \frac{1}{\tilde{c}_l} \sum_{i=\hat{k}}^{k} \|p_i - p_{i-1}\|^2$$

$$\overset{(37)}{\leq} \mathcal{L}_{\tilde{c}_l}(z_{\hat{k}}, p_{\hat{k}-1}) - \mathcal{L}_{\tilde{c}_l}(z_k, p_k) - \frac{\|p_k\|^2}{2\tilde{c}_l} + \frac{9(k-\hat{k})\kappa_p^2}{\tilde{c}_l}$$

$$= \left(\sum_{i=\hat{k}+1}^{k} \lambda_i \right) \Delta_k + \frac{9(k-\hat{k})\kappa_p^2}{\tilde{c}_l},$$

where the last equality follows from the definition of Δ_k in (16). Now, using the above bound and (28) we have:

$$\frac{\sum_{i=\hat{k}+1}^{k} \lambda_i \|w_i\|^2}{\sum_{i=\hat{k}+1}^{k} \lambda_i} \leq C_\sigma \left(\Delta_k + \frac{9(k-\hat{k})\kappa_p^2}{\tilde{c}_l \sum_{i=\hat{k}+1}^{k} \lambda_i} \right) \overset{(28)}{\leq} C_\sigma \left(\Delta_k + \frac{9\kappa_p^2}{\underline{\lambda} \tilde{c}_l} \right).$$

The result follows immediately from the above bound. □

The next result establishes bounds on $\|Az_k - b\|$ and on the quantity Δ_k defined in (16).

Lemma 3.5. *Consider the sequence of iterates $\{(z_k, c_k, p_k)\}_{k \in \mathcal{C}_l}$ generated during the l-th cycle of AS-PAL and let Δ_k be as in (16). Then, for every $k \in \mathcal{C}_l$,*

(a) we have

$$\|Az_k - b\| \leq \frac{2\kappa_p}{\tilde{c}_l}; \tag{38}$$

(b) if additionally $k \geq \hat{k}+1$, then

$$\Delta_k \leq \frac{\kappa_d}{\sum_{i=\hat{k}+1}^{k} \lambda_i}, \tag{39}$$

where κ_d is as in (21) and $\hat{k}$ denotes the first index in $\mathcal{C}_l$.

Proof. (a) Let $k \in \mathcal{C}_l$. Using the update for p_k in (15), triangle inequality, and the bound on p_k in (33), we have:

$$\|Az_k - b\| \overset{(15)}{=} \frac{\|p_k - p_{k-1}\|}{c_k} \overset{k \in \mathcal{C}_l}{\leq} \frac{\|p_k\| + \|p_{k-1}\|}{\tilde{c}_l} \overset{(33)}{\leq} \frac{2\kappa_p}{\tilde{c}_l}$$

which immediately proves (38).

(b) Recall from (17) that $\mathcal{C}_l := \{k : c_k = \tilde{c}_l := 2^{l-1}c_1\}$. Then, using the Cauchy-Schwarz inequality, the definition of the Lagrangian function in (2), the definition of S in (21), relations (33) and (38), and the fact that $\tilde{c}_l \geq c_1$, we have

$$\mathcal{L}_{\tilde{c}_l}(z_{\hat{k}}, p_{\hat{k}-1}) \leq S + \|p_{\hat{k}-1}\| \|Az_{\hat{k}} - b\| + \frac{\tilde{c}_l}{2} \|Az_{\hat{k}} - b\|^2 \overset{(38)}{\leq} S + \|p_{\hat{k}-1}\| \left(\frac{2\kappa_p}{\tilde{c}_l} \right) + \frac{2\kappa_p^2}{\tilde{c}_l} \overset{(33)}{\leq} S + \frac{4\kappa_p^2}{c_1}. \tag{40}$$

Let $k \in \mathcal{C}_l$ be such that $k \geq \hat{k}+1$. Using the definition of ϕ_* in (19) and completing the square, we have:

$$\mathcal{L}_{\tilde{c}_l}(z_k,p_k) - \phi_* \geq \mathcal{L}_{\tilde{c}_l}(z_k,p_k) - (f+h)(z_k) = \frac{1}{2}\left\|\frac{p_k}{\sqrt{\tilde{c}_l}} + \sqrt{\tilde{c}_l}(Az_k-b)\right\|^2 - \frac{\|p_k\|^2}{2\tilde{c}_l} \geq -\frac{\|p_k\|^2}{2\tilde{c}_l}. \quad (41)$$

Hence, it follows from the definition of Δ_k in (16) and relations (40) and (41) that

$$\Delta_k = \frac{1}{\sum_{i=\hat{k}+1}^k \lambda_i}\left(\mathcal{L}_{\tilde{c}_l}(z_{\hat{k}},p_{\hat{k}-1}) - \mathcal{L}_{\tilde{c}_l}(z_k,p_k) - \frac{\|p_k\|^2}{2\tilde{c}_l}\right) \leq \frac{1}{\sum_{i=\hat{k}+1}^k \lambda_i}\left(S + \frac{4\kappa_p^2}{c_1} - \phi_*\right).$$

Thus, (39) immediately follows from the definition of κ_d in (21). ☐

The following result establishes bounds on the number of ACG and outer iterations performed during an AS-PAL cycle and shows that AS-PAL outputs a $(\hat{\rho},\hat{\eta})$-approximate stationary solution of (1) within a logarithmic number of cycles.

Proposition 3.6. *The following statements about AS-PAL hold:*

(a) every cycle performs at most

$$\left\lceil 2 + \frac{2C_\sigma \kappa_d}{\underline{\lambda}\hat{\rho}^2}\right\rceil \quad (42)$$

outer iterations, where $\underline{\lambda}$, κ_d, *and* C_σ *are as in* (19), (21), *and* (10) *respectively; moreover, if* $\overline{\lambda}$ *is such that* $\overline{\lambda} = \Omega(m_f^{-1})$ *and* $\log_0^+(m_f\overline{\lambda}) \leq \mathcal{O}(1+\kappa_d/(\underline{\lambda}\hat{\rho}^2))$, *then the number of ACG calls within an arbitrary cycle is* $\mathcal{O}(1+m_f\kappa_d/\hat{\rho}^2)$*;*

(b) for any cycle l *of AS-PAL, its penalty parameter satisfies* $\tilde{c}_l \leq \max\{c_1, 2\hat{c}\}$ *where* $\hat{c} := \hat{c}(\hat{\rho},\hat{\eta})$ *and* $\hat{c}(\hat{\rho},\hat{\eta})$ *is as in* (20)*; as a consequence, the number of cycles of AS-PAL is bounded by*

$$\log_1^+\left(\frac{4\hat{c}}{c_1}\right) \quad (43)$$

where c_1 *is the initial penalty parameter for AS-PAL.*

Proof. (a) Fix a cycle l and let $\hat{k} = k_l$ denote the first index in $\mathcal{C}_l$ (see (18)). If some $k \in \mathcal{C}_l$ is such that

$$k > \hat{k} + \frac{2C_\sigma\kappa_d}{\underline{\lambda}\hat{\rho}^2} \quad (44)$$

then

$$\Delta_k \overset{(39)}{\leq} \frac{\kappa_d}{\sum_{i=\hat{k}+1}^k \lambda_i} \overset{(28)}{\leq} \frac{\kappa_d}{\underline{\lambda}(k-\hat{k})} \overset{(44)}{\leq} \frac{\hat{\rho}^2}{2C_\sigma} \quad (45)$$

which clearly implies that Δ_k satisfies inequality (16) and hence that the l-th cycle ends at or before the k-th iteration. Hence, the first part of (a) follows immediately from this conclusion. To prove the second part, first note that the number of times λ is divided by 2 in step 2 of AS-PAL is at most $\lceil \log_0^+(\overline{\lambda}/\underline{\lambda})/\log 2\rceil$, in view of the last conclusion of Proposition 3.1(b). This observation, the conclusion of the first part, the two conditions imposed on $\overline{\lambda}$, and the definition of $\underline{\lambda}$ in (19), then imply that the number of ACG calls within an arbitrary cycle is $\mathcal{O}(1+m_f\kappa_d/\hat{\rho}^2)$.

(b) Assume by contradiction that $\tilde{c}_l > \max\{c_1, 2\hat{c}\}$. Since $\tilde{c}_1 = c_1$ in view of (17), this implies that $l > 1$ and $\tilde{c}_l > 2\hat{c}$ and hence that $\tilde{c}_{l-1} > \hat{c}$ in view of the fact that $\tilde{c}_l = 2\tilde{c}_{l-1}$. Hence, it follows from the definition of $\hat{c} := \hat{c}(\hat{\rho},\hat{\eta})$ in (20) and Lemma 3.5(a) with $l = l-1$ that for every $k \in \mathcal{C}_{l-1}$,

$$\|Az_k - b\| \overset{(38)}{\leq} \frac{2\kappa_p}{\tilde{c}_{l-1}} < \frac{2\kappa_p}{\hat{c}} < \eta. \quad (46)$$

This implies that $\min_{i\in\mathcal{C}_{l-1}}\|w_i\| > \hat\rho$ in view of the termination criterion of step 3 and the fact that AS-PAL has not stopped in the $(l-1)$-th cycle. Letting $\hat k := k_{l-1}$, this conclusion together with Lemma 3.4 with $l = l-1$ then imply that

$$\hat\rho^2 < \frac{\sum_{i=\hat k+1}^{k}\lambda_i\|w_i\|^2}{\sum_{i=\hat k+1}^{k}\lambda_i} \le C_\sigma\left(\Delta_k + \frac{9\kappa_p^2}{\underline{\lambda}\tilde c_{l-1}}\right) < C_\sigma\left(\Delta_k + \frac{9\kappa_p^2}{\underline{\lambda}\hat c}\right) \le C_\sigma\Delta_k + \frac{\hat\rho^2}{2}$$

where the third inequality follows from the fact that $\tilde c_{l-1} > \hat c$ and the fourth one follows the definition of $\hat c := \hat c(\hat\rho,\hat\eta)$ in (20). Using this last conclusion, we can easily see that (16) is violated for every $k \in \mathcal{C}_{l-1}$ such that $k \ge \hat k + 1$, a conclusion that contradicts the fact that the $(l-1)$-th cycle terminated. □

We are now ready to prove Theorem 2.1.

of Theorem 2.1. First, note that the assumptions that $\overline{\lambda} = \Omega(m_f^{-1})$, $\log_0^+(m_f\overline{\lambda}) \le \mathcal{O}(1+\kappa_d/(\underline{\lambda}\hat\rho^2))$, the definition of $\underline{\lambda}$ in (19), and the second conclusion of Proposition 3.6(a) imply that every cycle of AS-PAL performs $\mathcal{O}(1+m_f\kappa_d/\hat\rho^2)$ ACG calls. Second, the assumption that $c_1 \le 4\hat c$ and Proposition 3.6(b) imply that $\tilde c_l \le 4\hat c$ and hence that $\mathcal{M}(\tilde c_l) \le \mathcal{M}(\hat c)$ in view of the definition of $\mathcal{M}(c)$ in Theorem 2.1. The result then immediately follows from the above observations, Proposition 3.1(a), and the bound (43) on the number of cycles performed by AS-PAL. □

4 Numerical Experiments

This section showcases the numerical performance of AS-PAL, nicknamed ASL, against five other benchmark algorithms for solving five classes of linearly-constrained SNCO problems. It contains five subsections. Each subsection reports the numerical results on a different class of linearly-constrained SNCO problems.

We have implemented a more aggressive variant of ASL, whose details we now describe. First, the variant differs from ASL in that it allows the prox stepsize to be doubled in step 5 of any iteration if it has not been halved in step 2 and the number of iterations performed by its ACG call in step 1 has not exceeded a pre-specified number. Second, since the prox stepsize is allowed to increase in this variant, the initial prox stepsize is taken to be relatively small. Third, our implementation chooses the following values for the input parameters of ASL:

$$\sigma = 0.1, \quad \mu = 1/4, \quad \chi = 0.5005, \quad \beta = 1.25, \quad p_0 = 0.$$

Finally, for $k \ge 1$, if L_k is the last estimated Lipschitz constant generated by ADAP-FISTA at the end of step 2 of the k^{th} iteration of ASL, then we take $M_0^{k+1} = L_k$.

Now, we describe the implementation details of the five benchmark algorithms which we compare our algorithm with. We consider the iALM method of [21], two variants of the S-prox-ALM of [37, 38] (nicknamed SPA1 and SPA2), the inexact proximal augmented Lagrangian method of [16] (nicknamed IPL), and the relaxed quadratic penalty method of [14] (nicknamed RQP). The implementation of iALM chooses the parameters σ, β_0, w_0, $\mathbf{y}^0$, and γ_k as

$$\sigma = 5, \quad \beta_0 = 1, \quad w_0 = 1, \quad \boldsymbol{y}^0 = 0, \quad \gamma_k = \frac{(\log 2)\,\|Ax^1\|}{(k+1)\,[\log(k+2)]^2},$$

for every $k \ge 1$. Furthermore, the implementation of iALM uses the ACG subroutine called APG. The starting point for the k^{th} APG call is the prox center for the k^{th} prox subproblem. The implementations of SPA1 and SPA2 also choose the parameters α, p, c, β, y_0, and z_0 as

$$\alpha = \frac{\Gamma}{4}, \quad p = 2(L_f + \Gamma\|A\|^2), \quad c = \frac{1}{2(L_f + \Gamma\|A\|^2)}, \quad \beta = 0.5, \quad y_0 = 0, \quad z_0 = x_0,$$

where $\Gamma = 1$ in SPA1 and $\Gamma = 10$ in SPA2. The implementation of IPL sets $\sigma = 0.3$, initial penalty parameter $c_1 = 1$, and constant prox parameter $\lambda = 1/(2m_f)$. RQP uses the AIPPv2 variant in [14] with initial prox stepsize $\lambda = 1/m_f$, $\sigma = 0.3$, and parameters $(\theta, \tau) = (4, 10\,[\lambda L_f + 1])$. Finally, note that IPL and RQP solve each prox subproblem using the ACG variant in [27] with an adaptive line search for the ACG variant's stepsize parameter as described in [10].

We describe the type of solution each of the methods aims to find. That is, given a linear operator A, functions f and h satisfying assumptions described in Subsection 2.1, an initial point $z_0 \in \mathcal{H}$, and tolerance pair $(\hat{\rho}, \hat{\eta}) \in \Re^2_{++}$, each of the methods aims to find a triple (z, p, w) satisfying:

$$w \in \nabla f(z) + \partial h(z) + A^* p, \quad \frac{\|w\|}{1 + \|\nabla f(z_0)\|} \le \hat{\rho}, \quad \frac{\|Az - b\|}{1 + \|Az_0 - b\|} \le \hat{\eta}, \tag{47}$$

where $\|\cdot\|$ signifies the Euclidean norm when solving vector problems and the Frobenius norm when solving matrix problems. Note that SPA1 and SPA2 are only included for comparison in the experiments of Subsection 4.1 and Subsection 4.2 since they are only guaranteed to converge when h is the indicator function of a polyhedron.

The tables below report the runtimes and the total number of ACG iterations needed to find a triple satisfying (47). The bold numbers in the tables of this section indicate the algorithm that performed the best for that particular metric (i.e. runtime or ACG iterations). It will be seen in the following subsections that the two adaptive methods ASL and RQP are the most consistent ones among the six considered. More specifically, within the specified time limit for each problem class, ASL converged in all instances considered in our experiments while RQP converged in 90% of them. To compare these two methods on a particular problem class more closely, we also report in each table caption the following average time ratio (ATR) between ASL and RQP defined as

$$ATR = \frac{1}{N} \sum_{i=1}^{N} a_i / r_i, \tag{48}$$

where N is the number of class instances that both methods were able to solve and a_i and r_i are the runtimes of ASL and RQP for instance i, respectively.

Finally, we note that all experiments were performed in MATLAB 2020a and run on a Macbook Pro with 8-core Intel Core i9 processor and 32 GB of memory. All codes for these experiments are also available online[2].

4.1 Nonconvex QP

Given a pair of dimensions $(\ell, n) \in \mathbb{N}^2$, a scalar pair $(\tau_1, \tau_2) \in \Re^2_{++}$, matrices $A, C \in \Re^{\ell \times n}$ and $B \in \Re^{n \times n}$, positive diagonal matrix $D \in \Re^{n \times n}$, and a vector pair $(b, d) \in \Re^\ell \times \Re^\ell$, we consider the problem

$$\begin{aligned} \min_z \quad & \left[f(z) := -\frac{\tau_1}{2}\|DBz\|^2 + \frac{\tau_2}{2}\|Cz - d\|^2 \right] \\ \text{s.t.} \quad & Az = b, \quad z \in \Delta^n, \end{aligned}$$

where $\Delta^n := \{x \in \Re^n_+ : \sum_{i=1}^n x_i = 1\}$.

Parameters		**Iteration Count**						**Runtime (seconds)**					
m_f	L_f	iALM	IPL	RQP	ASL	SPA1	SPA2	iALM	IPL	RQP	ASL	SPA1	SPA2
10^0	10^1	176005	11202	**3905**	4762	*	*	4410.98	373.54	**111.11**	232.89	*	*
10^0	10^2	109988	5382	6065	**1884**	*	*	1878.44	155.70	198.38	**64.50**	*	*
10^0	10^3	57869	1210	11216	**645**	*	*	1607.23	55.25	384.88	**20.39**	*	*
10^1	10^1	236655	3958	3171	**1236**	*	*	4785.86	144.45	88.10	**39.36**	*	*
10^1	10^2	195714	2319	6701	**1051**	*	*	3582.34	84.14	217.43	**34.07**	*	*
10^1	10^3	98865	1171	7583	**644**	*	*	2073.41	41.98	234.67	**20.55**	*	*
10^1	10^4	87595	6506	15637	**924**	*	*	3272.03	280.97	403.79	**29.51**	*	*
10^2	10^3	366178	*	7647	**778**	92872	*	6637.79	*	207.66	**25.22**	3290.91	*
10^2	10^4	248673	*	10421	**1375**	120882	257973	4329.35	*	283.96	**45.27**	5363.80	10644.96
10^2	10^5	130351	19887	16250	**2410**	205483	213369	2310.50	561.16	447.53	**80.98**	9317.19	7548.79
10^3	10^3	363915	*	4589	**2001**	*	*	8111.85	*	136.56	**71.45**	*	*
10^3	10^4	344723	*	6023	**4055**	*	158622	6949.95	*	596.66	**168.01**	*	6136.37
10^3	10^5	291006	16455	10067	**3007**	*	286333	5714.73	495.64	279.27	**107.44**	*	10761.87
10^3	10^6	141115	21586	15991	**2208**	269687	175752	2527.15	610.60	423.22	**89.73**	9718.92	6267.26

Table 1: Iteration counts and runtimes (in seconds) for the Nonconvex QP problem in Subsection 4.1. The tolerances are set to 10^{-4}. Entries marked with * did not converge in the time limit of 10800 seconds. The ATR metric is 0.3644.

Parameters		**Iteration Count**			**Runtime (seconds)**		
m_f	L_f	iALM	RQP	ASL	iALM	RQP	ASL
10^0	10^1	591803	23935	**8276**	9779.56	599.52	**419.69**
10^0	10^2	698270	62409	**2474**	11336.43	1579.43	**87.09**
10^0	10^3	551623	84314	**959**	9146.99	2232.40	**31.16**
10^1	10^1	*	25312	**1628**	*	703.17	**66.27**
10^1	10^2	*	53161	**1793**	*	3386.99	**77.04**
10^1	10^3	*	54172	**927**	*	1438.63	**34.01**
10^1	10^4	*	108376	**1477**	*	3482.75	**79.91**
10^2	10^3	*	92292	**1251**	*	2475.48	**61.80**
10^2	10^4	*	78775	**1992**	*	2116.42	**110.03**
10^2	10^5	*	137886	**3940**	*	3875.34	**219.18**
10^3	10^3	*	47491	**2238**	*	1280.58	**130.45**
10^3	10^4	*	49708	**6035**	*	596.66	**168.01**
10^3	10^5	*	52883	**3863**	*	1481.81	**220.99**
10^3	10^6	*	108743	**3396**	*	4083.58	**179.08**

Table 2: Iteration counts and runtimes (in seconds) for the Nonconvex QP problem in Subsection 4.1. The tolerances are set to 10^{-6}. Entries marked with * did not converge in the time limit of 21600 seconds. The ATR metric is 0.1173.

For our experiments in this subsection, we choose dimensions $(l, n) = (20, 1000)$ and generate the matrices A, B, and C to be fully dense. The entries of A, B, C, and d (resp. D) are generated by sampling from the uniform distribution $\mathcal{U}[0,1]$ (resp. $\mathcal{U}[1,1000]$). We generate the vector b as $b = A(\boldsymbol{e}/n)$ where $\boldsymbol{e}$ denotes the vector of all ones. The initial starting point z_0 is generated as $z^*/\sum_{i=1}^n z_i^*$, where the entries of z^* are sampled from the $\mathcal{U}[0,1]$ distribution. Finally, we choose

[2]See https://github.com/asujanani6/AS-PAL

$(\tau_1, \tau_2) \in \Re^2_{++}$ so that $L_f = \lambda_{\max}(\nabla^2 f)$ and $m_f = -\lambda_{\min}(\nabla^2 f)$ are the various values given in the tables of this subsection.

We now describe the specific parameters that ASL, RQP, and iALM choose for this class of problems. Both ASL and RQP choose the initial penalty parameter, $c_1 = 1$. ASL allows the prox stepsize to be doubled at the end of an iteration if the number of iterations by its ACG call does not exceed 75. ASL also takes M_0^1 defined in its step 1 to be 100 and the initial prox stepsize to be $20/m_f$. Finally, the auxillary parameters of iALM are given by:

$$B_i = \|a_i\|, \quad L_i = 0, \quad \rho_i = 0 \quad \forall i \geq 1,$$

where a_i is the i^{th} row of A.

The numerical results are presented in two tables, Table 1 and Table 2. The first table, Table 1, compares ASL with all five benchmark algorithms namely, iALM, IPL, RQP, SPA1, and SPA2. The tolerances are set as $\hat{\rho} = \hat{\eta} = 10^{-4}$ and a time limit of 10800 seconds, or 3 hours, is imposed. Table 2 presents the same exact instances as Table 1 but now with tolerances set as $\hat{\rho} = \hat{\eta} = 10^{-6}$ and a time limit of 21600 seconds, or 6 hours. Table 2 only compares ASL with iALM and RQP since these were the only two other algorithms to converge for every instance with tolerances set at 10^{-4}. Entries marked with * did not converge in the time limit.

4.2 Nonconvex QP with Box Constraints

Given a pair of dimensions $(\ell, n) \in \mathbb{N}^2$, a scalar triple $(r, \tau_1, \tau_2) \in \Re^3_{++}$, matrices $A, C \in \Re^{\ell \times n}$ and $B \in \Re^{n \times n}$, positive diagonal matrix $D \in \Re^{n \times n}$, and a vector pair $(b, d) \in \Re^\ell \times \Re^\ell$, we consider the problem

$$\begin{aligned}
&\min_z \ \left[f(z) := -\frac{\tau_1}{2}\|DBz\|^2 + \frac{\tau_2}{2}\|Cz - d\|^2 \right] \\
&\text{s.t. } Az = b, \\
&\qquad -r \leq z_i \leq r, \quad i \in \{1, ..., n\}.
\end{aligned}$$

For our experiments in this subsection, we choose dimensions $(l, n) = (20, 100)$ and generate the matrices A, B, and C to be fully dense. The entries of A, B, C, and d (resp. D) are generated by sampling from the uniform distribution $\mathcal{U}[0, 1]$ (resp. $\mathcal{U}[1, 1000]$). We generate the vector b as $b = A(u)$ where u is a random vector in $\mathcal{U}[-r, r]^n$. The initial starting point z_0 is generated as a random vector in $\mathcal{U}[-r, r]^n$. We vary r across the different instances. Finally, we choose $(\tau_1, \tau_2) \in \Re^2_{++}$ so that $L_f = \lambda_{\max}(\nabla^2 f)$ and $m_f = -\lambda_{\min}(\nabla^2 f)$ are the various values given in the tables of this subsection.

We now describe the specific parameters that ASL and RQP choose for this class of problems. Both ASL and RQP choose the initial penalty parameter, $c_1 = 1$. ASL also allows the prox stepsize to be doubled at the end of an iteration if the number of iterations by its ACG call does not exceed 75. Finally, ASL takes M_0^1 defined in its step 1 to be 100 and the initial prox stepsize to be $20/m_f$.

The numerical results are presented in Table 3. Table 3 compares ASL with all five of the benchmark algorithms namely, iALM, IPL, RQP, SPA1, and SPA2. The tolerances are set as $\hat{\rho} = \hat{\eta} = 10^{-5}$ and a time limit of 3600 seconds, or 1 hour, is imposed. Entries marked with * did not converge in the time limit.

Parameters			Iteration Count						Runtime (seconds)					
r	m_f	L_f	iALM	IPL	RQP	ASL	SPA1	SPA2	iALM	IPL	RQP	ASL	SPA1	SPA2
5	10^0	10^1	203310	11274	49512	**7247**	205576	1943184	226.93	17.93	92.43	**1.69**	335.91	2879.25
10	10^0	10^1	221433	9170	70736	**7043**	128567	1176352	334.57	14.29	132.59	**1.76**	240.07	2139.45
20	10^0	10^1	192970	8363	58980	**5469**	154035	1403641	307.75	14.59	144.06	**1.31**	374.16	2295.86
1	10^1	10^2	465159	*	326336	**4509**	133522	303003	858.38	*	1156.69	**1.17**	213.68	524.57
2	10^1	10^2	862136	*	399982	**8453**	64280	447451	1141.23	*	814.19	**2.01**	107.55	693.07
5	10^1	10^2	1857919	*	174005	**8320**	106715	488965	2476.33	*	394.47	**2.11**	238.12	879.75
1	10^1	10^3	351468	*	47007	**8438**	47583	123195	510.028	*	81.74	**2.00**	65.33	166.48
2	10^1	10^3	368578	*	69875	**6200**	96971	161433	481.14	*	129.77	**1.58**	123.39	198.84
5	10^1	10^3	280346	*	116988	**5218**	272448	161327	329.16	*	232.13	**1.24**	361.41	216.67
1	10^2	10^3	727587	*	104411	**4200**	*	112604	908.05	*	205.03	**1.15**	*	154.60
2	10^2	10^3	964734	21472	130903	**6432**	*	53266	1225.22	44.19	253.02	**1.56**	*	74.85
5	10^2	10^3	705884	11709	117945	**5137**	*	47237	890.93	25.93	226.21	**1.29**	*	65.84
1	10^2	10^4	576627	255622	100193	**7796**	155586	183307	864.79	575.34	200.17	**1.98**	232.28	274.50
2	10^2	10^4	1028921	29123	165257	**7048**	158192	196930	1477.99	57.82	314.62	**1.79**	256.01	308.35
5	10^2	10^4	652822	65523	86597	**9471**	144334	181157	1032.59	116.36	169.88	**2.22**	223.96	274.79
5	10^3	10^3	*	142961	225865	**26333**	*	*	*	253.35	439.62	**5.93**	*	*
10	10^3	10^3	2474551	*	168397	**14213**	*	*	3522.28	*	330.62	**3.27**	*	*
1	10^3	10^4	435881	71369	*	**4724**	*	*	667.19	154.75	*	**1.21**	*	*
2	10^3	10^4	476462	23931	64649	**8971**	*	*	584.73	39.52	100.27	**2.21**	*	*
5	10^3	10^4	521072	9829	*	**5943**	*	*	649.28	17.02	*	**1.51**	*	*
1	10^3	10^5	*	347105	*	**8952**	*	142702	*	696.61	*	**2.18**	*	231.41
2	10^3	10^5	1436029	*	*	**9013**	*	163317	2222.25	*	*	**2.20**	*	397.06
5	10^3	10^5	*	106935	*	**11629**	*	145047	*	276.73	*	**2.81**	*	192.72

Table 3: Iteration counts and runtimes (in seconds) for the Nonconvex QP problem with box constraints in Subsection 4.2. The tolerances are set to 10^{-5}. Entries marked with * did not converge in the time limit of 3600 seconds. The ATR metric is 0.0102.

4.3 Nonconvex QSDP

Given a pair of dimensions $(\ell, n) \in \mathbb{N}^2$, a scalar pair $(\tau_1, \tau_2) \in \Re^2_{++}$, linear operators $\mathcal{A} : \mathbb{S}^n_+ \mapsto \Re^\ell$, $\mathcal{B} : \mathbb{S}^n_+ \mapsto \Re^n$, and $\mathcal{C} : \mathbb{S}^n_+ \mapsto \Re^\ell$ defined by

$$[\mathcal{A}(Z)]_i = \langle A_i, Z \rangle, \quad [\mathcal{B}(Z)]_j = \langle B_j, Z \rangle, \quad [\mathcal{C}(Z)]_i = \langle C_i, Z \rangle,$$

for matrices $\{A_i\}_{i=1}^\ell, \{B_j\}_{j=1}^n, \{C_i\}_{i=1}^\ell \subseteq \Re^{n\times n}$, positive diagonal matrix $D \in \Re^{n\times n}$, and a vector pair $(b, d) \in \Re^\ell \times \Re^\ell$, we consider the following nonconvex quadratic semidefinite programming (QSDP) problem:

$$\begin{aligned} \min_{Z} \ & \left[f(Z) := -\frac{\tau_1}{2}\|D\mathcal{B}(Z)\|^2 + \frac{\tau_2}{2}\|\mathcal{C}(Z) - d\|^2 \right] \\ \text{s.t. } & \mathcal{A}(Z) = b, \quad Z \in P^n, \end{aligned}$$

where $P^n = \{Z \in \mathbb{S}^n_+ : \text{trace}\,(Z) = 1\}$.

For our experiments in this subsection, we choose dimensions $(l, n) = (30, 100)$. The matrices A_i, B_j, and C_i are generated so that only 5% of their entries are nonzero. The entries of A_i, B_j, C_i, and d (resp. D) are generated by sampling from the uniform distribution $\mathcal{U}[0, 1]$ (resp.

Parameters		Iteration Count				Runtime (seconds)			
m_f	L_f	iALM	IPL	RQP	ASL	iALM	IPL	RQP	ASL
10^0	10^1	230272	27345	9887	**9647**	1772.75	322.82	91.62	**85.38**
10^0	10^2	91421	4575	7085	**1498**	516.42	53.13	73.65	**13.28**
10^0	10^3	113405	1403	9486	**960**	587.24	19.54	112.42	**8.32**
10^0	10^4	393953	3140	10019	**1824**	1794.35	31.54	91.49	**15.98**
10^0	10^5	1938432	16282	15719	**9883**	9473.18	166.02	145.12	**85.50**
10^1	10^2	347506	*	15971	**4417**	1556.07	*	140.53	**38.31**
10^1	10^3	177264	*	10945	**2151**	750.97	*	96.93	**18.98**
10^1	10^4	129617	1296	9838	**1273**	2008.28	15.58	93.88	**11.14**
10^1	10^5	287924	3410	8040	**2262**	1305.24	35.51	75.28	**19.91**
10^1	10^6	1473676	15855	12696	**10305**	7865.52	164.07	120.96	**92.97**
10^2	10^4	182388	*	10803	**1261**	844.88	*	99.14	**11.55**
10^2	10^6	450561	4503	10804	**2990**	2612.55	59.18	128.30	**25.63**
10^2	10^7	1034041	20235	14622	**11893**	4612.17	207.29	137.38	**104.67**
10^3	10^4	552738	*	18530	**1368**	2435.47	*	172.45	**12.42**
10^3	10^5	220303	*	14929	**3543**	937.05	*	138.71	**31.34**
10^3	10^7	371617	5969	11230	**5121**	1791.77	56.92	96.31	**44.50**
10^3	10^8	1634409	23075	**13465**	17371	7250.46	245.84	**133.86**	149.43
10^4	10^5	450523	54984	18981	**4756**	1908.97	529.99	168.37	**42.61**
10^4	10^6	248709	*	15876	**6293**	1055.40	*	143.12	**54.94**
10^4	10^8	491118	7959	13184	**7187**	2230.07	83.00	125.98	**63.00**

Table 4: Iteration counts and runtimes (in seconds) for the Nonconvex QSDP problem in Subsection 4.3. The tolerances are set to 10^{-5}. Entries marked with * did not converge in the time limit of 10800 seconds. The ATR metric is 0.3831.

$\mathcal{U}[1, 1000]$). We generate the vector b as $b = \mathcal{A}(E/n)$, where E is the diagonal matrix in $\Re^{n\times n}$ with all ones on the diagonal. The initial starting point z_0 is generated as a random matrix in $\mathbb{S}^+_n$. The specific procedure for generating it is described in [17]. Finally, we choose $(\tau_1, \tau_2) \in \Re^2_{++}$ so that $L_f = \lambda_{\max}(\nabla^2 f)$ and $m_f = -\lambda_{\min}(\nabla^2 f)$ are the various values given in the tables of this subsection.

We now describe the specific parameters that ASL, RQP, and iALM choose for this class of problems. Both ASL and RQP choose the initial penalty parameter, $c_1 = 1$. ASL allows the prox stepsize to be doubled at the end of an iteration if the number of iterations by its ACG call does not exceed 75. ASL also takes M_0^1 defined in its step 1 to be 100 and the initial prox stepsize to be $1/(20m_f)$. Finally, the auxillary parameters of iALM are given by:

$$B_i = \|A_i\|_F, \quad L_i = 0, \quad \rho_i = 0 \quad \forall i \geq 1.$$

The numerical results are presented in two tables, Table 4 and Table 5. The first table, Table 4, compares ASL with three of the benchmark algorithms namely, iALM, IPL, and RQP. The tolerances are set as $\hat{\rho} = \hat{\eta} = 10^{-5}$ and a time limit of 10800 seconds, or 3 hours, is imposed. Table 5 presents the same exact instances as Table 4 but now with tolerances set as $\hat{\rho} = \hat{\eta} = 10^{-6}$ and a time limit of 14400 seconds, or 4 hours. Table 5 only compares ASL with iALM and RQP since these were the only two other algorithms to converge for every instance with tolerances set at 10^{-5}. Entries marked with * did not converge in the time limit.

Parameters		Iteration Count			Runtime (seconds)		
m_f	L_f	iALM	RQP	ASL	iALM	RQP	ASL
10^0	10^1	555086	35311	**15699**	2257.85	333.79	**138.34**
10^0	10^2	268608	27247	**2130**	1091.32	237.73	**18.24**
10^0	10^3	355922	26981	**2073**	1497.22	243.00	**17.59**
10^0	10^4	1317510	60908	**2453**	5523.22	563.85	**21.21**
10^0	10^5	*	70646	**10479**	*	699.03	**91.90**
10^1	10^2	1297322	68257	**7114**	5529.71	676.14	**61.71**
10^1	10^3	526262	41340	**24596**	2254.62	381.20	**212.61**
10^1	10^4	370204	35879	**2098**	1565.84	322.52	**18.06**
10^1	10^5	998029	42708	**3848**	4212.47	387.52	**32.55**
10^1	10^6	*	36575	**10710**	*	325.00	**90.22**
10^2	10^4	689898	39912	**1847**	2954.75	377.93	**15.83**
10^2	10^6	1345701	49506	**3658**	5725.54	448.12	**31.57**
10^2	10^7	*	43571	**12300**	*	399.37	**111.21**
10^3	10^4	1714445	64949	**1611**	7243.63	594.67	**13.94**
10^3	10^5	596094	40706	**3769**	2740.11	363.31	**32.95**
10^3	10^7	1625487	57454	**7867**	6691.03	511.35	**68.70**
10^3	10^8	*	45759	**18245**	*	399.79	**163.00**
10^4	10^5	1376159	*	**5030**	6145.45	*	**43.15**
10^4	10^6	995529	51540	**6552**	4392.18	489.43	**56.81**
10^4	10^8	1309587	72323	**8096**	5634.84	659.91	**71.30**

Table 5: Iteration counts and runtimes (in seconds) for the Nonconvex QSDP problem in Subsection 4.3. The tolerances are set to 10^{-6}. Entries marked with * did not converge in the time limit of 14400 seconds. The ATR metric is 0.1616.

4.4 Sparse PCA

We consider the sparse principal components analysis problem studied in [5]. That is, given integer k, positive scalar pair $(\vartheta, b) \in \Re^2_{++}$, and matrix $\Sigma \in S^n_+$, we consider the following sparse principal component analysis (PCA) problem:

$$\begin{aligned} \min_{\Pi,\Phi} \ & \langle \Sigma, \Pi \rangle_F + \sum_{i,j=1}^{n} q_\vartheta(\Phi_{ij}) + \vartheta \sum_{i,j=1}^{n} |\Phi_{ij}| \\ \text{s.t.} \ & \Pi - \Phi = 0, \quad (\Pi, \Phi) \in \mathcal{F}^k \times \Re^{n\times n} \end{aligned}$$

where $\mathcal{F}^k = \{M \in S^n_+ : 0 \preceq M \preceq I, \operatorname{tr} M = k\}$ denotes the k–Fantope and q_ϑ is the minimax concave penalty (MCP) function given by

$$q_\vartheta(t) := \begin{cases} -t^2/(2b), & \text{if } |t| \le b\vartheta, \\ b\vartheta^2/2 - \vartheta|t|, & \text{if } |t| > b\vartheta, \end{cases} \quad \forall t \in \Re.$$

For our experiments in this subsection, we choose $\vartheta = 100$ and allow b to vary. Observe that the curvature parameters are $m_f = L_f = 1/b$. We also generate the matrix Σ according to an eigenvalue decomposition $\Sigma = P\Lambda P^T$, based on a parameter pair (s, k), where k is as in the problem description and s is a positive integer. Specifically, we choose $\Lambda = (100, 1, ..., 1)$, the first column of P to be a sparse vector whose first s entries are $1/\sqrt{s}$, and the other entries of P to be sampled randomly from the standard Gaussian distribution. For our experiments, we fix $s = 5$ and allow k to vary. Also, for every problem instance, the initial starting point is chosen as $(\Pi_0, \Phi_0) = (D_k, 0)$ where D_k is a diagonal matrix whose first k entries are 1 and whose remaining entries are 0.

Parameters			**Iteration Count**				**Runtime (seconds)**			
k	m_f	L_f	iALM	IPL	RQP	ASL	iALM	IPL	RQP	ASL
5	125	125	*	1438	375618	**428**	*	14.84	2776.15	**2.32**
10	125	125	*	1559	40342	**546**	*	8.33	216.42	**2.87**
20	125	125	*	1400	13440	**442**	*	7.02	67.92	**2.39**
5	200	200	*	6555	21868	**773**	*	43.98	146.79	**4.25**
10	200	200	*	7470	10219	**894**	*	47.53	64.34	**5.70**
20	200	200	*	20132	60369	**612**	*	118.99	312.28	**4.25**
5	250	250	*	10391	19365	**622**	*	44.25	79.76	**3.96**
10	250	250	211991	32566	143192	**827**	574.91	175.81	690.92	**5.10**
20	250	250	236490	199353	*	**558**	628.55	985.49	*	**3.22**
5	1000	1000	*	567358	63633	**1818**	*	2873.66	312.34	**11.97**
10	1000	1000	*	*	*	**932**	*	*	*	**14.59**
20	1000	1000	*	*	*	**1581**	*	*	*	**12.62**

Table 6: Iteration counts and runtimes (in seconds) for the Sparse PCA problem in Subsection 4.4. The tolerances are set to 10^{-5}. Entries marked with * did not converge in the time limit of 3600 seconds. The ATR metric is 0.0306.

We now describe the specific parameters that ASL, RQP, and iALM choose for this class of problems. Both ASL and RQP choose the initial penalty parameter, $c_1 = 1$. ASL allows the prox stepsize to be doubled at the end of an iteration if the number of iterations by its ACG call does not exceed 4. ASL also takes M_0^1 defined in its step 1 to be 1 and the initial prox stepsize to be $1/(2m_f)$. Finally, the auxillary parameters of iALM are chosen as:

$$B_i = 1, \quad L_i = 0, \quad \rho_i = 0 \quad \forall i \geq 1,$$

based off the relaxed, but unverified assumption that its iterates lie in $\mathcal{F}^k \times \mathcal{F}^k$.

The numerical results are presented in Table 6. Table 6 compares ASL with three of the benchmark algorithms namely, iALM, IPL, and RQP. The tolerances are set as $\hat{\rho} = \hat{\eta} = 10^{-5}$ and a time limit of 3600 seconds, or 1 hour, is imposed. Entries marked with * did not converge in the time limit.

4.5 Bounded Matrix Completion (BMC)

We consider the bounded matrix completion problem studied in [35]. That is, given a dimension pair $(p, q) \in \mathbb{N}^2$, positive scalar triple $(\upsilon, \tau_m, \theta) \in \Re^3_{++}$, scalar pair $(u, l) \in \Re^2$, matrix $Q \in \Re^{p\times q}$, and indices Ω, we consider the following bounded matrix completion (BMC) problem:

$$\begin{aligned} \min_X \ & \frac{1}{2}\|P_\Omega(X - Q)\|^2 + \tau_m \sum_{i=1}^{\min\{p,q\}} [\kappa(\sigma_i(X)) - \kappa_0\sigma_i(X)] + \tau_m\kappa_0\|X\|_* \\ \text{s.t. } & l \leq X_{ij} \leq u \quad \forall (i,j) \in \{1, ..., p\} \times \{1, ..., q\}, \end{aligned}$$

where $\|\cdot\|_*$ denotes the nuclear norm, the function P_Ω is the linear operator that zeros out any entry not in Ω, the function $\sigma_i(X)$ denotes the i^{th} largest singular value of X, and

$$\kappa_0 := \frac{\upsilon}{\theta}, \quad \kappa(t) := \upsilon \log\left(1 + \frac{|t|}{\theta}\right) \quad \forall t \in \Re.$$

Parameters				**Iteration Count**		**Runtime (seconds)**	
θ	τ_m	m_f	L_f	RQP	ASL	RQP	ASL
1/2	0.5	2	2	130	**79**	209.07	**139.52**
1/2	1	4	4	128	**119**	207.21	**189.75**
1/2	2	8	8	1233	**457**	2075.16	**931.91**
1/3	0.5	4.5	4.5	384	**51**	1229.60	**61.22**
1/3	1	9	9	513	**76**	1360.69	**101.04**
1/3	2	18	18	*	**494**	*	**1001.82**
1/4	0.5	8	8	601	**66**	928.01	**85.28**
1/4	1	16	16	680	**90**	1077.89	**147.76**
1/5	0.5	12.5	12.5	488	**193**	1653.75	**313.51**
1/5	1	25	25	859	**227**	1494.94	**475.09**
1/6	0.5	18	18	838	**96**	1359.45	**137.72**
1/6	1	36	36	617	**221**	962.52	**358.25**
1/7	0.5	24.5	24.5	770	**142**	1232.90	**195.66**
1/7	1	49	49	789	**355**	1213.75	**580.04**

Table 7: Iteration counts and runtimes (in seconds) for the BMC problem in Subsection 4.5. The tolerances are set to 10^{-3}. Entries marked with * did not converge in the time limit of 7200 seconds. The ATR metric is 0.3079.

We first describe the parameters considered for the above problem and some of its properties. First, the matrix Q is the user-movie ratings data matrix of the MovieLens 100K dataset[3]. Second, υ is chosen to be 0.5 and τ_m and θ are allowed to vary. Third, the curvature parameters are $m_f = 2\upsilon\tau_m/\theta^2$ and $L_f = \max\{1, m_f\}$. Fourth, the bounds are set to $(l, u) = (0, 5)$ and the initial starting point is chosen as $X_0 = 0$. Finally, the above optimization problem can be written in the form:

$$\begin{aligned} \min_X \ & f(X) + h(X) \\ \text{s.t.} \ & \mathcal{A}(X) \in S, \end{aligned}$$

where

$$f(X) = \frac{1}{2}\|P_\Omega(X - Q)\|^2 + \tau_m \sum_{i=1}^{\min\{p,q\}} \left[\kappa(\sigma_i(X)) - \kappa_0\sigma_i(X)\right], \quad h(X) = \tau_m\kappa_0\|X\|_*,$$
$$\mathcal{A}(X) = X, \quad S = \left\{Z \in \Re^{p\times q} : l \le Z_{ij} \le u, \ (i,j) \in \{1, ..., p\} \times \{1, ..., q\}\right\}.$$

[3]See the MovieLens 100K dataset containing 610 users and 9724 movies which can be found in https://grouplens.org/datasets/movielens/

To deal with the more generalized constraints $\mathcal{A}(X) \in S$, our method, ASL, considers the following augmented Lagrangian function and Lagrange multiplier update:

$$\mathcal{L}_c(z,p) := f(z) + h(z) - \frac{\|p\|^2}{2c} + \frac{c}{2}\|(Az + \frac{p}{c}) - \Pi_S(Az + \frac{p}{c})\|^2;$$
$$p_k := p_{k-1} + c_k(Az_k - \Pi_S(Az_k + \frac{p_{k-1}}{c_k})),$$

where Π_S denotes the projection onto the set S. We only compare ASL to the RQP method since RQP was the only other method developed and coded to deal with these generalized constraints.

We now describe the specific parameters that ASL and RQP choose for this class of problems. Both ASL and RQP choose the initial penalty parameter, $c_1 = 500$. ASL allows the prox stepsize to be doubled at the end of an iteration if the number of iterations by its ACG call does not exceed 4. Finally, ASL also takes M_0^1 defined in its step 1 to be 1 and the initial prox stepsize to be $10/(m_f)$.

The numerical results are presented in Table 7. Table 7 compares ASL with RQP. The tolerances are set as $\hat{\rho} = \hat{\eta} = 10^{-3}$ and a time limit of 7200 seconds, or 2 hours, is imposed. Entries marked with * did not converge in the time limit.

4.6 Comments about the numerical results

Overall, the two adaptive methods ASL and RQP were the most reliable and consistent, converging in almost every instance. ASL was clearly the most efficient, often converging much faster than RQP particularly when the required accuracy was high. As demonstrated by the results in Tables 1 and 2, and the ones in Tables 4 and 5, the ATR metric improves (decreases) as the required accuracy increases. Finally, ASL worked extremely fast on the problem classes of Subsections 4.2 and 4.4 as demonstrated by the results in Tables 3 and 6, respectively.

A ADAP-FISTA algorithm

A.1 ADAP-FISTA method

This subsection presents an adaptive ACG variant, called ADAP-FISTA, which is an important tool in the development of the AS-PAL method. We first introduce the assumptions on the problem it solves. ADAP-FISTA considers the following problem

$$\min\{\psi(x) := \psi_s(x) + \psi_n(x) : x \in \Re^n\} \tag{49}$$

where ψ_s and ψ_n are assumed to satisfy the following assumptions:

(I) $\psi_n : \Re^n \to \Re \cup \{+\infty\}$ is a possibly nonsmooth convex function;

(II) $\psi_s : \Re^n \to \Re$ is a differentiable function and there exists $\bar{L} \geq 0$ such that

$$\|\nabla\psi_s(z') - \nabla\psi_s(z)\| \leq \bar{L}\|z' - z\| \quad \forall z, z' \in \Re^n. \tag{50}$$

We now describe the type of approximate solution that ADAP-FISTA aims to find.

Problem A: Given ψ satisfying the above assumptions, a point $x_0 \in \operatorname{dom}\psi_n$, a parameter $\sigma \in (0,\infty)$, the problem is to find a pair $(y,u) \in \operatorname{dom}\psi_n \times \Re^n$ such that

$$\|u\| \leq \sigma\|y - x_0\|, \quad u \in \nabla\psi_s(y) + \partial\psi_n(y). \tag{51}$$

We are now ready to present the ADAP-FISTA algorithm below.

ADAP-FISTA Method

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0. Let initial point $x_0 \in \operatorname{dom}\psi_n$ and scalars $\mu > 0$, $L_0 > \mu$, $\chi \in (0,1)$, $\beta > 1$, and $\sigma > 0$ be given, and set $y_0 = x_0$, $A_0 = 0$, $\tau_0 = 1$, and $j = 0$;

1. Set $L_{j+1} = L_j$;

2. Compute

$$a_j = \frac{\tau_j + \sqrt{\tau_j^2 + 4\tau_j A_j (L_{j+1} - \mu)}}{2(L_{j+1} - \mu)}, \quad \tilde{x}_j = \frac{A_j y_j + a_j x_j}{A_j + a_j}, \tag{52}$$

$$y_{j+1} := \underset{u \in \mathrm{dom}\,\psi_n}{\operatorname{argmin}} \left\{ q_j(u; \tilde{x}_j, L_{j+1}) := \ell_{\psi_s}(u; \tilde{x}_j) + \psi_n(u) + \frac{L_{j+1}}{2}\|u - \tilde{x}_j\|^2 \right\}, \tag{53}$$

If the inequality

$$\ell_{\psi_s}(y_{j+1}; \tilde{x}_j) + \frac{(1-\chi)L_{j+1}}{2}\|y_{j+1} - \tilde{x}_j\|^2 \geq \psi_s(y_{j+1}) \tag{54}$$

holds go to step 3; else set $L_{j+1} \leftarrow \beta L_{j+1}$ and repeat step 2;

3. Compute

$$A_{j+1} = A_j + a_j, \quad \tau_{j+1} = \tau_j + a_j \mu, \tag{55}$$

$$s_{j+1} = (L_{j+1} - \mu)(\tilde{x}_j - y_{j+1}), \tag{56}$$

$$x_{j+1} = \frac{1}{\tau_{j+1}}\left[\mu a_j y_{j+1} + \tau_j x_j - a_j s_{j+1}\right]; \tag{57}$$

4. If the inequality

$$\|y_{j+1} - x_0\|^2 \geq \chi A_{j+1} L_{j+1} \|y_{j+1} - \tilde{x}_j\|^2, \tag{58}$$

holds, then go to step 5; otherwise, stop with **failure**;

5. Compute

$$u_{j+1} = \nabla\psi_s(y_{j+1}) - \nabla\psi_s(\tilde{x}_j) + L_{j+1}(\tilde{x}_j - y_{j+1}). \tag{59}$$

If the inequality

$$\|u_{j+1}\| \leq \sigma \|y_{j+1} - x_0\| \tag{60}$$

holds then stop with **success** and output $(y, u) := (y_{j+1}, u_{j+1})$; otherwise, $j \leftarrow j + 1$ and go to step 1.

We now make some remarks about ADAP-FISTA. First, usual FISTA methods for solving the strongly convex version of (49) consist of repeatedly invoking only steps 2 and 3 of ADAP-FISTA either with a static Lipschitz constant, namely, $L_{j+1} = L$ for all $j \geq 0$ for some $L \geq \bar{L}$, or by adaptively searching for a suitable Lipschitz L_{j+1} (as in step 2 of ADAP-FISTA) satisfying a condition similar to (54). Second, the pair (y_{j+1}, u_{j+1}) always satisfies the inclusion in (51) (see Lemma A.3 below) so if ADAP-FISTA stops successfully in step 5, or equivalently (60) holds, the pair solves Problem A above. Finally, if condition (58) in step 4 is never violated, ADAP-FISTA must stop successfully in step 5 (see Proposition A.1 below).

We now discuss how ADAP-FISTA compares with existing ACG variants for solving (49) under the assumption that ψ_s is μ-strongly convex. Under this assumption, FISTA variants have been studied, for example, in [3, 11, 12, 27, 29], while other ACG variants have been studied, for example, in [7, 8, 30]. A crucial difference between ADAP-FISTA and these variants is that: i) ADAP-FISTA stops based on a different relative criterion, namely, (60) (see Problem A above) and attempts to approximately solve (49) in this sense even when ψ_s is not μ-strongly convex, and ii) ADAP-FISTA provides a key and easy to check inequality whose validity at every iteration guarantees its successful termination. On the other hand, ADAP-FISTA shares similar features with these other methods in that: i) it has a reasonable iteration complexity guarantee regardless of whether it succeeds or fails, and ii) it successfully terminates when ψ_s is μ-strongly convex (see Propositions A.1-A.2 below). Moreover, like the method in [3], ADAP-FISTA adaptively searches for a suitable Lipschitz estimate L_{j+1} that is used in (53).

We now present the main convergence results of ADAP-FISTA, which is invoked by AS-PAL for solving the sequence of subproblems (3). The first result, namely Proposition A.1 below, gives an iteration complexity bound regardless if ADAP-FISTA terminates with success or failure and shows that if ADAP-FISTA successfully stops, then it obtains a stationary solution of (49) with respect to a relative error criterion. The second result, namely Proposition A.2 below, shows that ADAP-FISTA always stops successfully whenever ψ_s is μ-strongly convex.

Proposition A.1. *The following statements about ADAP-FISTA hold:*

(a) if $L_0 = \mathcal{O}(\bar{L})$, it always stops (with either success or failure) in at most

$$\mathcal{O}_1\left(\sqrt{\frac{\bar{L}}{\mu}}\log_0^+(\bar{L})\right)$$

iterations/resolvent evaluations;

(b) if it stops successfully, it terminates with a pair $(y,u) \in \operatorname{dom}\psi_n \times \Re^n$ *satisfying*

$$u \in \nabla\psi_s(y) + \partial\psi_n(y); \tag{61}$$

$$\|u\| \le \sigma\|y - x_0\|. \tag{62}$$

Proposition A.2. *If ψ_s is μ-convex, then ADAP-FISTA always terminates with success and its output (y,u), in addition to satisfying (61) and (62), also satisfies the inclusion $u \in \partial(\psi_s + \psi_n)(y)$.*

The rest of this section is broken up into two subsections which are dedicated to proving Proposition A.1 and Proposition A.2, respectively.

A.2 Proof of Proposition A.1

This subsection is dedicated to proving Proposition A.1. The first lemma below presents key definitions and inequalities used in the convergence analysis of ADAP-FISTA.

Lemma A.3. *Define*

$$\omega = \beta/(1-\chi), \quad \zeta := \bar{L} + \max\{L_0, \omega\bar{L}\}. \tag{63}$$

Then, the following statements hold:

(a) $\{L_j\}$ is nondecreasing;

(b) for every $j \ge 0$, we have

$$\tau_j = 1 + A_j\mu, \quad \frac{\tau_j A_{j+1}}{a_j^2} = L_{j+1} - \mu; \tag{64}$$

$$L_0 \le L_j \le \max\{L_0, \omega\bar{L}\}; \tag{65}$$

$$u_{j+1} \in \nabla\psi_s(y_{j+1}) + \partial\psi_n(y_{j+1}), \quad \|u_{j+1}\| \le \zeta\|y_{j+1} - \tilde{x}_j\|. \tag{66}$$

Proof. (a) It is clear from the update rule in the beginning of Step 1 that $\{L_j\}$ is nondecreasing.

(b) The first equality in (64) follows directly from both of the relations in (55). The second equality in (64) follows immediately from the definition of a_j in (52) and the first relation in (55).

We prove (65) by induction. It clearly holds for $j = 0$. Suppose now (65) holds for $j \ge 0$ and let us show that it holds for $j+1$. Note that if $L_{j+1} = L_j$, then relation (65) immediately holds. Assume then that $L_{j+1} > L_j$. It then follows from the way L_{j+1} is chosen in step 1 that (54) is not satisfied with L_{j+1}/β. This fact together with the inequality (50) at the points $(y_{j+1}, \tilde{x}_j)$ imply that

$$\ell_{\psi_s}(y_{j+1};\tilde{x}_j) + \frac{(1-\chi)L_{j+1}}{2\beta}\|y_{j+1} - \tilde{x}_j\|^2 < \psi_s(y_{j+1}) \overset{(50)}{\le} \ell_{\psi_s}(y_{j+1};\tilde{x}_j) + \frac{\bar{L}}{2}\|y_{j+1} - \tilde{x}_j\|^2. \tag{67}$$

The relation in (65) then immediately follows from the definition of ω in (63).

Now, by the definition of u_{j+1} in (59), triangle inequality, (50), the bound (65) on L_{j+1}, and the definition of ζ we have

$$\frac{\|u_{j+1}\|}{\|y_{j+1} - \tilde{x}_j\|} \overset{(59)}{\le} \frac{\|\nabla\psi_s(y_{j+1}) - \nabla\psi_s(\tilde{x}_j)\|}{\|y_{j+1} - \tilde{x}_j\|} + L_{j+1} \overset{(50)}{\le} \bar{L} + L_{j+1} \overset{(65)}{\le} \zeta$$

which immediately implies the inequality in (66). It follows from (53) and its associated optimality condition that $0 \in \nabla\psi_s(\tilde{x}_j) + \partial\psi_n(y_{j+1}) - L_{j+1}(\tilde{x}_j - y_{j+1})$, which in view of the definition of u_{j+1} in (59) implies the inclusion in (66). □

The result below gives some estimates on the sequence $\{A_j\}$, which will be important for the convergence analysis of the method.

Lemma A.4. *Define*

$$Q := 2\sqrt{\frac{\max\{L_0, \omega\bar{L}\}}{\mu}} \tag{68}$$

where ω is as in (63). Then, for every $j \ge 1$, we have

$$A_j L_j \ge \max\left\{\frac{j^2}{4}, \left(1 + Q^{-1}\right)^{2(j-1)}\right\}. \tag{69}$$

Proof. Let integer $j \geq 1$ be given. Define $\xi_j = 1/(L_j - \mu)$. Using the first equality in (55) and the definition of a_j in (52), we have that for every $i \leq j$,

$$A_i \overset{(55)}{=} A_{i-1} + a_{i-1} \overset{(52)}{\geq} A_{i-1} + \left(\frac{\tau_{i-1}\xi_i}{2} + \sqrt{\tau_{i-1}\xi_i A_{i-1}}\right) \geq \left(\sqrt{A_{i-1}} + \frac{1}{2}\sqrt{\tau_{i-1}\xi_i}\right)^2.$$

Passing the above inequality to its square root and using Lemma A.3(a) and the fact that (64) implies that $\tau_{i-1} \geq \max\{1, \mu A_{i-1}\}$, we then conclude that for every $i \leq j$,

$$\sqrt{A_i} - \sqrt{A_{i-1}} \geq \frac{1}{2}\sqrt{\xi_i} \geq \frac{1}{2}\sqrt{\xi_j} \tag{70}$$

$$\sqrt{\frac{A_i}{A_{i-1}}} \geq 1 + \frac{1}{2}\sqrt{\mu\xi_i} \geq 1 + \frac{1}{2}\sqrt{\mu\xi_j} \geq 1 + Q^{-1} \tag{71}$$

where the last inequality in (71) follows from the definition of ξ_j, the relation in (65), and the definition of Q in (68). Adding the inequality in (70) from $i = 1$ to $i = j$ and using the fact that $A_0 = 0$, we conclude that $\sqrt{A_j} \geq j\sqrt{\xi_j}/2$ and hence that the first bound in (69) holds in view of the fact that $\xi_j \geq 1/L_j$. Now, multiplying the inequality in (71) from $i = 2$ to $i = j$ and using Lemma A.3(a) and the fact that $A_1 = \xi_1$, we conclude that $\sqrt{A_j} \geq \sqrt{\xi_1}(1 + Q^{-1})^{j-1} \geq \sqrt{\xi_j}(1 + Q^{-1})^{j-1}$, and hence that the second bound in (69) holds in view of the fact that $\xi_j \geq 1/L_j$. □

Proposition A.5. *Let ζ and Q be as in (63) and (68), respectively. ADAP-FISTA always stops (with either success or failure) and does so by performing at most*

$$\left\lceil (1 + Q)\log_0^+\left(\frac{\zeta^2}{\chi\sigma^2}\right) + 1\right\rceil + \left\lceil \frac{\log_0^+(\bar{L}/((1-\chi)L_0))}{\log\beta}\right\rceil \tag{72}$$

iterations/resolvent evaluations.

Proof. Let l denote the first quantity in (72). Using this definition and the inequality $\log(1 + \alpha) \geq \alpha/(1 + \alpha)$ for any $\alpha > -1$, it is easy to verify that

$$\left(1 + Q^{-1}\right)^{2(l-1)} \geq \frac{\zeta^2}{\chi\sigma^2}. \tag{73}$$

We claim that ADAP-FISTA terminates with success or failure in at most l iterations. Indeed, it suffices to show that if ADAP-FISTA has not stopped with failure up to (and including) the l-th iteration, then it must stop successfully at the l-th iteration. So, assume that ADAP-FISTA has not stopped with failure up to the l-th iteration. In view of step 4 of ADAP-FISTA, it follows that (58) holds with $j = l - 1$.

This observation together with the inequality in (66) with $j = l - 1$, (69) with $j = l$, and (73), then imply that

$$\|y_l - x_0\|^2 \overset{(58)}{\geq} \chi A_l L_l \|y_l - \tilde{x}_{l-1}\|^2 \overset{(66)}{\geq} \frac{\chi}{\zeta^2} A_l L_l \|u_l\|^2 \overset{(69)}{\geq} \frac{\chi}{\zeta^2}\left(1 + Q^{-1}\right)^{2(l-1)} \|u_l\|^2 \overset{(73)}{\geq} \frac{1}{\sigma^2}\|u_l\|^2, \tag{74}$$

and hence that (60) is satisfied. In view of Step 5 of ADAP-FISTA, the method must successfully stop at the end of the l-th iteration. We have thus shown that the above claim holds. Moreover, in view of (65), it follows that the second term in (72) is a bound on the total number of times L_j is multiplied by β and step 2 is repeated. Since exactly one resolvent evaluation occurs every time step 2 is executed, the desired conclusion follows. □

We are now ready to give the proof of Proposition A.1.

of Proposition A.1. (a) The result immediately follows from Proposition A.5 and the assumption that $L_0 = \mathcal{O}(\bar{L})$.

(b) This is immediate from the termination criterion (60) in step 5 of ADAP-FISTA and the inclusion in (66). □

A.3 Proof of Proposition A.2

This subsection is dedicated to proving Proposition A.2. Thus, for the remainder of this subsection, assume that ψ_s is μ-strongly convex. The first lemma below presents important properties of the iterates generated by ADAP-FISTA.

Lemma A.6. *For every $j \geq 0$ and $x \in \Re^n$, define*

$$\gamma_j(x) := \ell_{\psi_s}(y_{j+1}, \tilde{x}_j) + \psi_n(y_{j+1}) + \langle s_{j+1}, x - y_{j+1}\rangle + \frac{\mu}{2}\|y_{j+1} - \tilde{x}_j\|^2 + \frac{\mu}{2}\|x - y_{j+1}\|^2, \tag{75}$$

where $\psi := \psi_s + \psi_n$ and s_{j+1} are as in (49) and (56), respectively. Then, for every $j \geq 0$, we have:

$$y_{j+1} = \underset{x}{\operatorname{argmin}} \left\{ \gamma_j(x) + \frac{L_{j+1} - \mu}{2} \|x - \tilde{x}_j\|^2 \right\}; \tag{76}$$

$$x_{j+1} = \underset{x \in \Re^n}{\operatorname{argmin}} \left\{ a_j \gamma_j(x) + \tau_j \|x - x_j\|^2 /2 \right\}. \tag{77}$$

Proof. Since $\nabla \gamma_j(y_{j+1}) = s_{j+1}$, it follows from (56) that y_{j+1} satisfies the optimality condition for (76), and thus the relation in (76) follows. Furthermore, we have that:

$$\begin{aligned} a_j \nabla \gamma_j(x_{j+1}) + \tau_j(x_{j+1} - x_j) &= a_j s_{j+1} + a_j \mu (x_{j+1} - y_{j+1}) + \tau_j (x_{j+1} - x_j) \\ &\overset{(55)}{=} a_j s_{j+1} - \mu a_j y_{j+1} - \tau_j x_j + \tau_{j+1} x_{j+1} \overset{(57)}{=} 0 \end{aligned}$$

and thus (77) follows. □

Before stating the next lemma, recall that if a closed function $\Psi : \Re^n \to \Re \cup \{+\infty\}$ is ν-convex with modulus $\nu > 0$, then it has an unique global minimum z^* and

$$\Psi(z^*) + \frac{\nu}{2}\| \cdot - z^*\|^2 \leq \Psi(\cdot). \tag{78}$$

Lemma A.7. *For every $j \geq 0$ and $x \in \Re^n$, we have*

$$\begin{aligned} A_j \gamma_j(y_j) + a_j \gamma_j(x) &+ \frac{\tau_j}{2}\|x_j - x\|^2 - \frac{\tau_{j+1}}{2}\|x_{j+1} - x\|^2 \\ &\geq A_{j+1}\psi(y_{j+1}) + \frac{\chi A_{j+1} L_{j+1}}{2}\|y_{j+1} - \tilde{x}_j\|^2. \end{aligned} \tag{79}$$

Proof. Using (77), the second identity in (55), and the fact that $\Psi_j := a_j \gamma_j(\cdot) + \tau_j \| \cdot - x_j\|^2/2$ is $(\tau_j + \mu a_j)$-convex, it follows from (78) with $\Psi = \Psi_j$ and $\nu = \tau_{j+1}$ that

$$a_j \gamma_j(x) + \frac{\tau_j}{2}\|x - x_j\|^2 - \frac{\tau_{j+1}}{2}\|x - x_{j+1}\|^2 \geq a_j \gamma_j(x_{j+1}) + \frac{\tau_j}{2}\|x_{j+1} - x_j\|^2 \quad \forall x \in \Re^n.$$

Using the convexity of γ_j, the definitions of A_{j+1} and $\tilde{x}_j$ in (55) and (52), respectively, and the second equality in (64), we have

$$\begin{aligned} A_j \gamma_j(y_j) &+ a_j \gamma_j(x_{j+1}) + \frac{\tau_j}{2}\|x_{j+1} - x_j\|^2 \\ &\geq A_{j+1}\gamma_j\left(\frac{A_j y_j + a_j x_{j+1}}{A_{j+1}}\right) + \frac{\tau_j A_{j+1}^2}{2a_j^2}\left\|\frac{A_j y_j + a_j x_{j+1}}{A_{j+1}} - \frac{A_j y_j + a_j x_j}{A_{j+1}}\right\|^2 \\ &\overset{(52)}{\geq} A_{j+1} \min_x \left[\gamma_j(x) + \frac{\tau_j A_{j+1}}{2a_j^2}\|x - \tilde{x}_j\|^2\right] \\ &\overset{(64)}{=} A_{j+1} \min_x \left\{\gamma_j(x) + \frac{L_{j+1} - \mu}{2}\|x - \tilde{x}_j\|^2\right\} \\ &\overset{(76)}{=} A_{j+1}\left[\gamma_j(y_{j+1}) + \frac{L_{j+1} - \mu}{2}\|y_{j+1} - \tilde{x}_j\|^2\right] \\ &\overset{(75)}{=} A_{j+1}\left[\ell_{\psi_s}(y_{j+1}; \tilde{x}_j) + \psi_n(y_{j+1}) + \frac{L_{j+1}}{2}\|y_{j+1} - \tilde{x}_j\|^2\right] \\ &\overset{(54)}{\geq} A_{j+1}\left[\psi(y_{j+1}) + \frac{\chi L_{j+1}}{2}\|y_{j+1} - \tilde{x}_j\|^2\right]. \end{aligned}$$

The conclusion of the lemma now follows by combining the above two relations. □

Lemma A.8. *For every $j \geq 0$, we have $\gamma_j \leq \psi$.*

Proof. Define:

$$\tilde{\gamma}_j(x) := \ell_{\psi_s}(x;\tilde{x}_j) + \psi_n(x) + \frac{\mu}{2}\|x-\tilde{x}_j\|^2. \tag{80}$$

It follows immediately from the fact that ψ_s is μ-convex that $\tilde{\gamma}_j \le \psi$. Furthermore, immediately from the definition of y_{j+1} in (53), we can write:

$$y_{j+1} = \underset{x}{\operatorname{argmin}} \left\{ \tilde{\gamma}_j(x) + \frac{L_{j+1}-\mu}{2}\|x-\tilde{x}_j\|^2 \right\}. \tag{81}$$

Now, clearly from (81) and the definition of s_{j+1} in (56), we see that $s_{j+1} \in \partial\tilde{\gamma}_j(y_{j+1})$. Furthermore, since $\tilde{\gamma}_j$ is μ-convex, it follows from the subgradient rule for the sum of convex functions that the above inclusion is equivalent to $s_{j+1} \in \partial\left(\tilde{\gamma}_j(\cdot) - \frac{\mu}{2}\|\cdot - y_{j+1}\|^2\right)(y_{j+1})$. Hence, the subgradient inequality and the fact that $\tilde{\gamma}_j(x) \le \psi(x)$ imply that for all $x \in \Re^n$:

$$\psi(x) \ge \tilde{\gamma}_j(x) \ge \tilde{\gamma}_j(y_{j+1}) + \langle s_{j+1}, x - y_{j+1}\rangle + \frac{\mu}{2}\|x-y_{j+1}\|^2 = \gamma_j(x)$$

and thus the statement of the lemma follows. □

Lemma A.9. *For every $j \ge 0$ and $x \in \operatorname{dom}\psi_n$, we have*

$$\eta_j(x) - \eta_{j+1}(x) \ge \frac{\chi A_{j+1}L_{j+1}}{2}\|y_{j+1}-\tilde{x}_j\|^2$$

where

$$\eta_j(x) := A_j[\psi(y_j) - \psi(x)] + \frac{\tau_j}{2}\|x-x_j\|^2.$$

Proof. Subtracting $A_{j+1}\psi(x)$ from both sides of the inequality in (79) and using Lemma A.8 we have

$$\begin{aligned} A_j\psi(y_j) + a_j\psi(x) - A_{j+1}\psi(x) + \frac{\tau_j}{2}\|x_j - x\|^2 - \frac{\tau_{j+1}}{2}\|x_{j+1}-x\|^2 \\ \ge A_{j+1}\psi(y_{j+1}) - A_{j+1}\psi(x) + \frac{\chi A_{j+1}L_{j+1}}{2}\|y_{j+1}-\tilde{x}_j\|^2. \end{aligned}$$

The result now follows from the first equality in (55) and the definition of $\eta_j(x)$. □

We now state a result that will be important for deriving complexity bounds for ADAP-FISTA.

Lemma A.10. *For every $j \ge 0$ and $x \in \operatorname{dom}\psi_n$, we have*

$$A_j[\psi(y_j) - \psi(x)] + \frac{\tau_j}{2}\|x-x_j\|^2 \le \frac{1}{2}\|x-x_0\|^2 - \frac{\chi}{2}\sum_{i=0}^{j-1} A_{i+1}L_{i+1}\|y_{i+1}-\tilde{x}_i\|^2. \tag{82}$$

Proof. Summing the inequality of Lemma A.9 from $j = 0$ to $j = j-1$, using the facts that $A_0 = 0$ and $\tau_0 = 1$, and using the definition of $\eta_j(\cdot)$ in Lemma A.9 gives us the inequality of the lemma. □

We are now ready to give the proof of Proposition A.2.

of Proposition A.2. Since ψ_s is μ-convex, Lemma A.10 holds. Thus, using (82) with $x = y_j$, it follows that for all $j \ge 0$:

$$\|y_j - x_0\|^2 \overset{(82)}{\ge} \chi\sum_{i=1}^{j} A_iL_i\|y_i - \tilde{x}_{i-1}\|^2 \ge \chi A_jL_j\|y_j - \tilde{x}_{j-1}\|^2. \tag{83}$$

Hence, for all $j \ge 0$, relation (58) in step 4 of ADAP-FISTA is always satisfied and thus ADAP-FISTA never fails. In view of this observation and Proposition A.1, it follows that if ψ_s is μ-convex then ADAP-FISTA always terminates successfully with a (y, u) satisfying relations (61) and (62) in a finite number of iterations. The inclusion $u \in (\psi_s + \psi_n)(y)$ then follows immediately from the inclusion in (61) and the subgradient rule for the sum of convex functions. □

B Technical Results for Proof of Lagrange Multipliers

The following basic result is used in Lemma B.3. Its proof can be found, for instance, in [4, Lemma A.4]. Recall that ν_A^+ denotes the smallest positive singular value of a nonzero linear operator A .

Lemma B.1. *Let $A : \Re^n \to \Re^l$ be a nonzero linear operator. Then,*

$$\nu_A^+ \|u\| \le \|A^* u\|, \quad \forall u \in A(\Re^n).$$

The following technical result, whose proof can be found in Lemma 3.10 of [16], plays an important role in the proof of Lemma B.3 below.

Lemma B.2. *Let h be a function as in (A1). Then, for every $\delta \ge 0$, $z \in \mathcal{H}$, and $\xi \in \partial_\delta h(z)$, we have*

$$\|\xi\| \text{dist}(u, \partial\mathcal{H}) \le [\text{dist}(u, \partial\mathcal{H}) + \|z - u\|]\, M_h + \langle \xi, z - u \rangle + \delta \quad \forall u \in \mathcal{H} \tag{84}$$

where $\partial\mathcal{H}$ denotes the boundary of $\mathcal{H}$.

Lemma B.3. *Assume that h is a function as in condition (A1) and $A : \Re^n \to \Re^l$ is a linear operator satisfying condition (A2). Assume also that the triple $(z, q, r) \in \Re^n \times A(\Re^n) \times \Re^n$ satisfy $r \in \partial h(z) + A^* q$. Then:*

(a) there holds

$$\bar{d}\nu_A^+ \|q\| \le 2D_h\left(M_h + \|r\|\right) - \langle q, Az - b \rangle; \tag{85}$$

(b) if, in addition,

$$q = q^- + \chi(Az - b) \tag{86}$$

for some $q^- \in \Re^l$ and $\chi > 0$, then we have

$$\|q\| \le \max\left\{ \|q^-\|, \frac{2D_h(M_h + \|r\|)}{\bar{d}\nu_A^+} \right\}. \tag{87}$$

Proof. (a) The assumption on (z, q, r) implies that $r - A^* q \in \partial h(z)$. Hence, using the Cauchy-Schwarz inequality, the definitions of $\bar{d}$ and $\bar{z}$ in (19) and (A2), respectively, and Lemma B.2 with $\xi = r - A^* q$, $u = \bar{z}$, and $\delta = 0$, we have:

$$\bar{d}\|r - A^* q\| - \left[\bar{d} + \|z - \bar{z}\|\right] M_h \overset{(84)}{\le} \langle r - A^* q, z - \bar{z} \rangle \le \|r\| \|z - \bar{z}\| - \langle q, Az - b \rangle. \tag{88}$$

Now, using the above inequality, the triangle inequality, the definition of D_h in (A1), and the facts that $\bar{d} \le D_h$ and $\|z - \bar{z}\| \le D_h$, we conclude that:

$$\bar{d}\|A^* q\| + \langle q, Az - b \rangle \overset{(88)}{\le} \left[\bar{d} + \|z - \bar{z}\|\right] M_h + \|r\| \left(D_h + \bar{d}\right) \le 2D_h\left(M_h + \|r\|\right). \tag{89}$$

Noting the assumption that $q \in A(\Re^n)$, inequality (85) now follows from the above inequality and Lemma B.1.

(b) Relation (86) implies that $\langle q, Az - b \rangle = \|q\|^2/\chi - \langle q^-, q \rangle/\chi$, and hence that

$$\bar{d}\nu_A^+ \|q\| + \frac{\|q\|^2}{\chi} \le 2D_h(M_h + \|r\|) + \frac{\langle q^-, q \rangle}{\chi} \le 2D_h(M_h + \|r\|) + \frac{\|q\|}{\chi}\|q^-\|, \tag{90}$$

where the last inequality is due to the Cauchy-Schwarz inequality. Now, letting K denote the right hand side of (87) and using (90), we conclude that

$$\left(\bar{d}\nu_A^+ + \frac{\|q\|}{\chi} \right) \|q\| \overset{(90)}{\le} \left(\frac{2D_h(M_h + \|r\|)}{K} + \frac{\|q\|}{\chi} \right) K \le \left(\bar{d}\nu_A^+ + \frac{\|q\|}{\chi} \right) K, \tag{91}$$

and hence that (87) holds. □

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