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Tuning of centrifugal pendulum vibration absorbers for translational and rotational vibration reduction

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ABSTRACT

This note provides an analytical proof of the optimal tuning of centrifugal pendulum vibration absorbers (CPVAs) to reduce in-plane translational and rotational vibration for a rotor with N cyclically symmetric substructures attached to it. The reaction forces that the substructures (helicopter or wind turbine blades, for example) exert on the rotor are first analyzed. The linearized equations of motion for the vibration are then solved by a gyroscopic system modal analysis procedure. The solutions show that the rotor translational vibrations are reduced when one group of CPVAs is tuned to order $N-1$ and another group is tuned to order $N+1$. Derivation of this result is not available in the literature. The current derivation also yields the better known result that tuning CPVAs to order N reduces rotational rotor vibration.

1. Introduction

Centrifugal pendulum vibration absorbers (CPVAs) are applied in rotating systems to counteract the vibration at a frequency proportional to the rotor speed. The constant of proportionality is called the order of rotation speed and is generally a rational number. The order is defined as the tuning order of the CPVAs. An absorber that is designed to work at a particular order of rotation speed is said to be tuned to that excitation order. In rotating systems with N attached cyclically periodic substructures, CPVAs that are tuned to the two orders $N \pm 1$ have been used to counteract rotor in-plane translational vibration. Bramwell et al. [1] described this feature of bifilar absorbers [2–4]. Bauchau et al. [5] numerically showed the effectiveness of translational vibration reduction in the four-bladed rotor of a Sikorsky UH-60 helicopter using absorbers tuned to the third or fifth order. Fig. 1 shows four cyclically symmetric CPVAs used in a four-bladed helicopter rotor to reduce the translational vibration of the rotor. Although this property has been used in helicopters, its proof and explanation is not available in the literature. A derivation and physical explanation of the property is useful for further analyses and for widening the application scope.

This work presents an analytical derivation of how in-plane translational vibration reduction of a rotor having N cyclically periodic substructures mounted to it is best achieved using CPVAs. The in-plane forces that the periodic substructures exert on the rotor are first analyzed in the rotating and fixed reference frames. The resulting rotor vibration amplitude is then derived. Given this excitation and the known modal properties of CPVA systems [6,7], we show that optimal in-plane translational vibration reduction occurs when two different groups of absorbers are used with one group tuned to order $N-1$ and the other to $N+1$. This result is generalized for cases where higher harmonics of the substructure forces are the most important. Secondarily, we also derive the known result that absorbers tuned to the Nth order reduce the rotor rotational vibration at that order [8–10].

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2. Rotors with cyclically periodic substructures

Fig. 2 shows the bases (unit vectors) and coordinates used in the rotor–substructure system. We consider N cyclically periodic substructures ($k = 1, 2, \ldots, N$) attached to the rotor. $\{\mathbf{E}_1, \mathbf{E}_2\}$ is the fixed basis. The rotor-fixed basis $\{\mathbf{e}_1^0, \mathbf{e}_2^0\}$ rotates with the rotor at constant mean speed Ω. The in-plane translational displacements of the rotor along the $\mathbf{e}_1^0$ and $\mathbf{e}_2^0$ directions are denoted by x and y, respectively. The coordinate μ describes the rotational vibration of the rotor relative to its nominal angle Ωt. The position of the kth substructure is defined by a spacing angle $\psi_k = 2(k-1)i\,/\,N$ as shown in Fig. 2. All angles for substructures and absorbers attached to the rotor are relative to the $\mathbf{e}_1^0$ axis. Thus, the first substructure with $\psi_1 = 0$ lies along $\mathbf{e}_1^0$.

The reaction forces that the kth substructure exerts on the rotor in the radial and tangential directions local to the kth substructure are denoted by R_{1k} and R_{2k}, respectively, as shown in Fig. 2. Transforming these reaction forces to the rotor-fixed basis $\{\mathbf{e}_1^0, \mathbf{e}_2^0\}$ yields

$$F_{x_k} = R_{1k}\cos\psi_k + R_{2k}\sin\psi_k, \tag{1a}$$

$$F_{y_k} = R_{1k}\sin\psi_k - R_{2k}\cos\psi_k. \tag{1b}$$

For steady rotor speed, the reaction forces are assumed to be periodic with fundamental frequency equal to the rotor speed. They can be expanded in Fourier series as

$$R_{1k}(t) = P_{0k} + \sum_{i=1}^{\infty} P_{ik}\cos[i(\Omega t + \psi_k)] + \sum_{i=1}^{\infty} Q_{ik}\sin[i(\Omega t + \psi_k)], \tag{2a}$$

$$R_{2k}(t) = S_{0k} + \sum_{i=1}^{\infty} S_{ik}\cos[i(\Omega t + \psi_k)] + \sum_{i=1}^{\infty} T_{ik}\sin[i(\Omega t + \psi_k)]. \tag{2b}$$

For identical, cyclically periodic substructures, the reaction force functions have the same shape, but different phase, for each substructure. The phase differences are captured in Eqs. (2a) and (2b) by the spacing angles ψ_k of the substructures. This is because when the rotor rotates an angle ψ_k, the kth substructure advances to the position previously occupied by the $(k+1)$st substructure. For identical substructures where the only differences in reaction force are from the angular position (or phase), the Fourier coefficients in Eqs. (2a) and (2b) are independent of k. Thus, $P_{0k} = P_0$ and $P_{ik} = P_i$ for $k = 1, 2, \ldots, N$ with similar expressions for Q_{ik}, S_{0k}, S_{ik}, and T_{ik}.

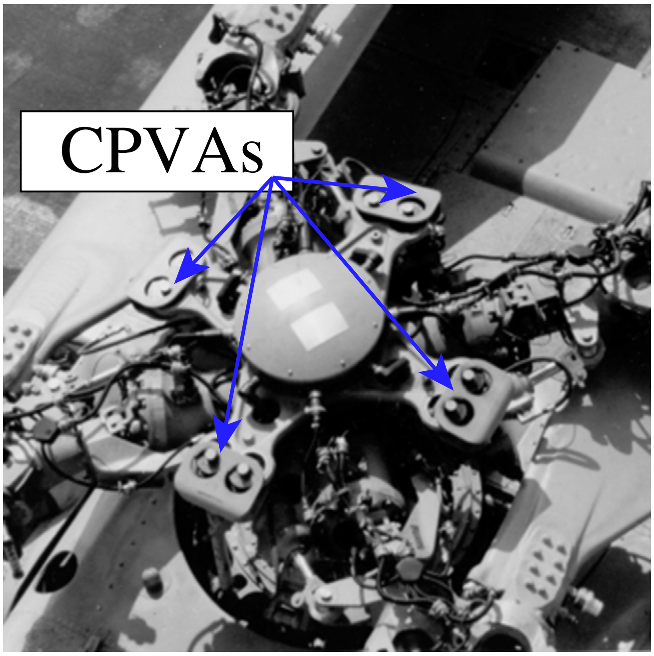

Fig. 1. Four cyclically symmetric CPVAs used for translational vibration reduction in a four-bladed helicopter rotor. These absorbers utilize a bifilar suspension with rollers, to allow for non-circular paths of the absorber center of mass [2].
This photo is taken by Steven W. Shaw on board the USS Nimitz in 1991.

From Eqs. (1a), (1b), (2a) and (2b), the total force in the $\mathbf{e}_1^0$ direction from all substructures is

$$\begin{aligned} F_x &= \sum_{k=1}^{N} F_{x_k} \\ &= \sum_{k=1}^{N} P_0 \cos\psi_k + \sum_{k=1}^{N} S_0 \sin\psi_k \\ &\quad + \sum_{k=1}^{N}\sum_{i=1}^{\infty} P_i \cos[i(\Omega t+\psi_k)]\cos\psi_k + \sum_{k=1}^{N}\sum_{i=1}^{\infty} Q_i \sin[i(\Omega t+\psi_k)]\cos\psi_k \\ &\quad + \sum_{k=1}^{N}\sum_{i=1}^{\infty} S_i \cos[i(\Omega t+\psi_k)]\sin\psi_k + \sum_{k=1}^{N}\sum_{k=1}^{\infty} T_i \sin[i(\Omega t+\psi_k)]\sin\psi_k \\ &= \frac{1}{2}\left\{ \sum_{i=1}^{\infty} [(P_i - T_i)\cos(i\Omega t) + (Q_i + S_i)\sin(i\Omega t)] \sum_{k=1}^{N} \cos[(i+1)\psi_k] \right. \\ &\quad + \sum_{i=1}^{\infty} [(P_i + T_i)\cos(i\Omega t) + (Q_i - S_i)\sin(i\Omega t)] \sum_{k=1}^{N} \cos[(i-1)\psi_k] \\ &\quad + \sum_{i=1}^{\infty} [(T_i - P_i)\sin(i\Omega t) + (Q_i + S_i)\cos(i\Omega t)] \sum_{k=1}^{N} \sin[(i+1)\psi_k] \\ &\quad \left. + \sum_{i=1}^{\infty} [(Q_i - S_i)\cos(i\Omega t) - (P_i + T_i)\sin(i\Omega t)] \sum_{k=1}^{N} \sin[(i-1)\psi_k] \right\}. \end{aligned} \tag{3}$$

The constant P_0 and S_0 terms vanish because $\sum_{k=1}^{N} cos\psi_k = \sum_{k=1}^{N} sin\psi_k = 0$ for cyclically periodic structures. For any i that is not an integer multiplier of N, $\sum_{k=1}^{N} cos(i\psi_k) = \sum_{k=1}^{N} sin(i\psi_k) = 0$ [6]. Thus, the first term of Eq. (3) survives only when $i + 1 = jN$, where j is a

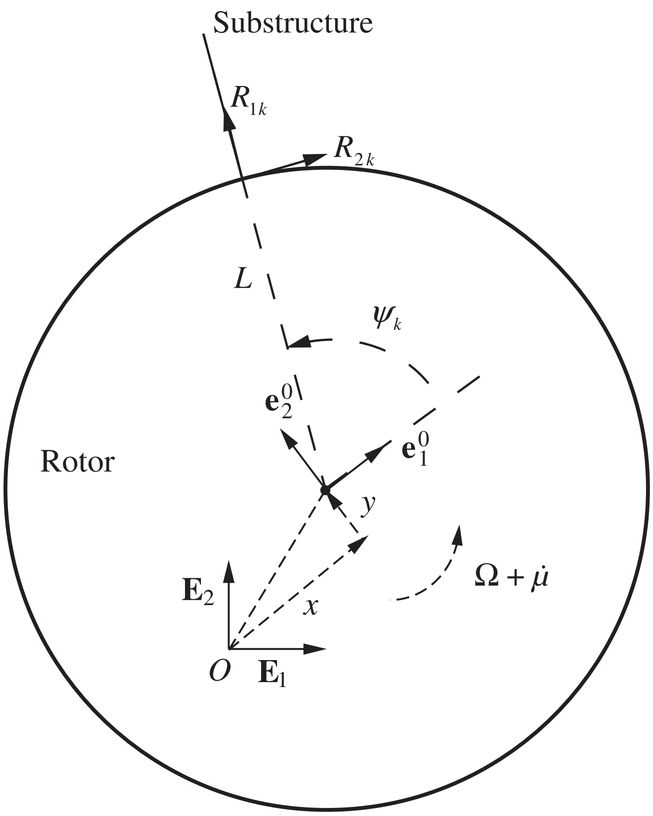

Fig. 2. Bases and coordinates used in the rotor–substructure system.

positive integer. The second term of Eq. (3) survives only when $i-1=jN$. The third and fourth terms of Eq. (3) vanish for all i. Additionally, $\sum_{k=1}^{N} cos[(i+1)\psi_k]=N$ for $i+1=jN$ and $\sum_{k=1}^{N} cos[(i-1)\psi_k]=N$ for $i-1=jN$. Thus, the force F_x becomes

$$\begin{aligned} F_x = \frac{N}{2}\sum_{j=1}^{\infty}\Big\{ & \left(P_{jN-1}-T_{jN-1}\right)\cos[(jN-1)\Omega t]+\left(Q_{jN-1}+S_{jN-1}\right)\sin[(jN-1)\Omega t] \\ & +\left(P_{jN+1}+T_{jN+1}\right)\cos[(jN+1)\Omega t]+\left(Q_{jN+1}-S_{jN+1}\right)\sin[(jN+1)\Omega t]\Big\}. \end{aligned} \quad (4)$$

Similarly,

$$\begin{aligned} F_y = \frac{N}{2}\sum_{j=1}^{\infty}\Big\{ & \left(P_{jN-1}-T_{jN-1}\right)\sin[(jN-1)\Omega t]-\left(Q_{jN-1}+S_{jN-1}\right)\cos[(jN-1)\Omega t] \\ & +\left(Q_{jN+1}-S_{jN+1}\right)\cos[(jN+1)\Omega t]-\left(P_{jN+1}+T_{jN+1}\right)\sin[(jN+1)\Omega t]\Big\}. \end{aligned} \quad (5)$$

According to Eqs. (4) and (5), the forces F_x and F_y have equal amplitudes. The frequencies of the periodic excitation forces F_x and F_y are $(jN-1)\Omega$ and $(jN+1)\Omega$ for $j=1, 2, \ldots, \infty$. The first two terms of F_x corresponding to frequency $(jN-1)\Omega$ are $-90°$ out-of-phase relative to the first two terms of F_y; the last two terms of F_x corresponding to frequency $(jN+1)\Omega$ are $90°$ out-of-phase relative to the last two terms of F_y.

The total torque that the substructures exert on the rotor is

$$\begin{aligned} T &= -L\sum_{k=1}^{N}R_{2k} \\ &= -\left\{NLS_0+L\sum_{k=1}^{N}\sum_{i=1}^{\infty}S_i\cos[i(\Omega t+\psi_k)]+L\sum_{k=1}^{N}\sum_{i=1}^{\infty}T_i\sin[i(\Omega t+\psi_k)]\right\} \\ &= -\left\{NLS_0+L\sum_{i=1}^{\infty}[S_i\cos(i\Omega t)+T_i\sin(i\Omega t)]\sum_{k=1}^{N}\cos(i\psi_k)\;\;+L\sum_{i=1}^{\infty}[T_i\cos(i\Omega t)-S_i\sin(i\Omega t)]\sum_{k=1}^{N}\sin(i\psi_k)\right\}, \end{aligned} \quad (6)$$

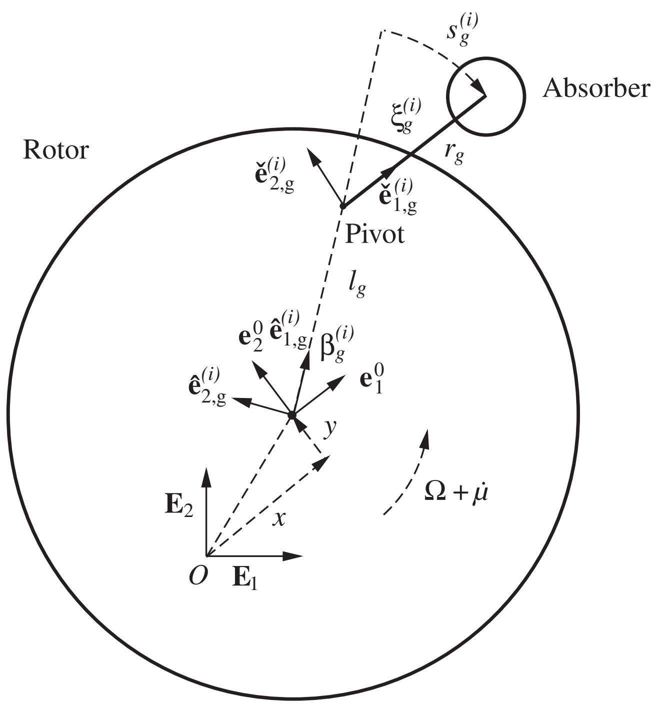

Fig. 3. Bases and coordinates used in the rotor–absorber system [7].

where L is the radius where the substructure forces act on the rotor. Because $\sum_{k=1}^{N}\cos(i\psi_k)=\sum_{k=1}^{N}\sin(i\psi_k)=0$ when i is not an integer multiplier of N [6], the second term in Eq. (6) survives only when $i = jN$ with $\sum_{k=1}^{N} cos(i\psi_k)=N$. The third term vanishes for all i. Thus, Eq. (6) reduces to

$$T=-NL\left\{S_0+\sum_{j=1}^{\infty}\left[S_{jN}\cos(jN\Omega t)+T_{jN}\sin(jN\Omega t)\right]\right\}. \tag{7}$$

The frequencies of the periodic excitation torque are $jN\Omega$.

Similar algebraic reductions resulting from cyclic symmetry occur for the net force and moment that the sun-planet contact forces exert on the sun gear of a planetary gear [11].

3. Rotors with centrifugal pendulum vibration absorbers

We consider that the rotor is fitted with p groups of CPVAs, where group g consists of N_g equally spaced, identical CPVAs. The CPVAs are bifilar absorbers [2–4] that dynamically act like point masses moving along a path fixed to the rotor, as shown schematically in Fig. 3. The angle $\beta_g^{(i)}$ denotes the pivot position (relative to $\mathbf{e}_1^0$) of the ith absorber in the gth group. The coordinate $s_g^{(i)}$ is the deviation of the ith absorber in the gth group along its path from the nominal position. The dimension l_g is the distance between the center of the rotor and the pivot, and r_g is the radius of the locally circular path followed by each absorber in the gth group. The mass of each absorber in the gth group is m_g. The total number of absorbers used in the system is denoted by $N_a=\sum_{g=1}^{p}N_g$. The mass and moment of inertia of the rotor are m_r and J_r (where J_r does not include the inertia of the absorbers). The rotor bearing stiffness, which is the same in the x- and y-directions, is denoted by k_r.

The linearized matrix equation of motion of the rotor–absorber system is derived in [6,7] and has the form

$$\mathbf{M}\ddot{\mathbf{q}}+\Omega\mathbf{G}\dot{\mathbf{q}}+\left(\mathbf{K}_b-\Omega^2\mathbf{K}_\Omega\right)\mathbf{q}=\mathbf{F}, \tag{8}$$

where the generalized coordinate vector is

$$\mathbf{q}=\left(\mathbf{q}_r,\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_p\right)^T, \tag{9a}$$

$$\mathbf{q}_r=(x,y,\mu)^T, \tag{9b}$$

$$\mathbf{q}_g=\left(s_g^{(1)},s_g^{(2)},\ldots,s_g^{(N_g)}\right)^T, \qquad g=1,2,\ldots,p. \tag{9c}$$

The external force vector is

$$\mathbf{F}=\left(F_x,F_y,T,\underbrace{0,0,\cdots,0}_{N_a}\right)^T, \tag{10}$$

because external loads only act on the rotor. The eigenvalue problem associated with Eq. (8) is

$$\lambda^2\mathbf{M}\phi+\lambda\Omega\mathbf{G}\phi+\left(\mathbf{K}_b-\Omega^2\mathbf{K}_\Omega\right)\phi=0. \tag{11}$$

There are only three mode types for the rotor–absorber system [6,7]: rotational, translational, and absorber modes. The rotational modes contain pure rotation (no translation) of the rotor, and the translational modes contain pure translation (no rotation) of the rotor. No rotor motion exists in the absorber modes.

Eq. (8) has the form of a standard gyroscopic system that is commonly solved by a state space method [12–14]. The state space formulation of Eq. (8) is

$$\mathbf{Az}+\mathbf{Bz}=\left(\mathbf{F}\quad 0_{1\times(N_a+3)}\right)^T, \qquad \mathbf{z}=(\mathbf{q}\quad \mathbf{q})^T, \tag{12a}$$

$$\mathbf{A}=\begin{pmatrix}\mathbf{M} & 0_{(N_a+3)\times(N_a+3)}\\ \text{symmetric} & \mathbf{K}_b-\Omega^2\mathbf{K}_\Omega\end{pmatrix}, \tag{12b}$$

$$\mathbf{B} = \begin{pmatrix} \Omega\mathbf{G} & \mathbf{K}_b - \Omega^2\mathbf{K}_\Omega \\ \text{skew–symmetric} & 0_{(N_a+3)\times(N_a+3)} \end{pmatrix}. \tag{12c}$$

The associated eigenvalue problem from $\mathbf{z} = \mathbf{u}e^{\lambda t}$ is $\lambda\mathbf{Au} + \mathbf{Bu} = 0$. The eigensolutions occur as complex conjugate pairs. With the symmetry properties of $\mathbf{A}$ and $\mathbf{B}$, the eigenvector orthogonality relations are $\overline{\mathbf{U}}^T\mathbf{AU} = \text{diag}\left(0, 0, \underbrace{1, 1, \ldots, 1}_{2N_a+4}\right)$ and $\overline{\mathbf{U}}^T\mathbf{BU} = -\text{diag}(0, 0, \lambda_2, \overline{\lambda}_2, \ldots, \lambda_{N_a+3}, \overline{\lambda}_{N_a+3})$, where $\mathbf{U}$ is a matrix with each column being $\mathbf{u}_i = (\lambda_i\phi_i \quad \phi_i)^T$ and ϕ_i is the complex-valued eigenvector of Eq. (11) corresponding to the eigenvalue λ_i. The first two diagonal zeros of the two orthogonality relations come from the pair of rigid body rotational modes with $\lambda_1 = \overline{\lambda}_1 = 0$.

The solution of Eqs. (12a), (12b) and (12c) is $\mathbf{z} = \sum_{i=1}^{N_a+3} (a_i(t)\mathbf{u}_i + \overline{a}_i(t)\overline{\mathbf{u}}_i) = \mathbf{Ua}(t)$, where $\mathbf{a}(t)$ is a vector of modal coordinates. Substitution of this solution into Eqs. (12a), (12b) and (12c) and pre-multiplication by $\overline{\mathbf{U}}^T$ yield the following uncoupled modal equations of motion for $i = 2, 3, \ldots, N_a + 3$

$$\begin{aligned} \dot{a}_i - \lambda_i a_i &= \overline{\lambda}_i \overline{\phi}_i^T \mathbf{F} \\ &= \begin{cases} \overline{\lambda}_i\left(\overline{x}_i F_x + \overline{y}_i F_y\right); & \text{if } i \text{ is a translational mode} \\ \overline{\lambda}_i \overline{\mu}_i T; & \text{if } i \text{ is a rotational mode } (i \neq 1), \\ 0; & \text{if } i \text{ is an absorber mode} \end{cases} \end{aligned} \tag{13}$$

where x_i denotes the rotor x-displacement of the ith eigenvector (and similarly for y_i and μ_i). This diagonalization by modal analysis gives a trivial equation for the rigid body rotational mode associated with a_1, and thus the rigid body mode response must be treated separately.

Because only translational modes have rotor translation, the rotor translational vibration responses $x(t)$ and $y(t)$ are determined solely by response of these modes. For these two motions, one only needs to consider the modal coordinates $a_i(t)$ associated with translational modes. Similarly, the rotor rotational vibration $\mu(t)$ is determined solely by the rotational modes.

For a given translational mode, the two rotor translational modal deflections x_i and y_i in Eq. (13) are either 90° or − 90° out-of-phase with equal amplitude [6,7]. The translational modes whose two rotor translational deflections are − 90° out-of-phase are the complex conjugate of those whose two rotor translational deflections are 90° out-of-phase. As noted earlier, the substructure forces F_x and F_y separate into two components that are 90° and − 90° out-of-phase between the corresponding F_x and F_y components. The corresponding F_x and F_y components also have equal amplitudes. These relationships dictate which force components can excite which modes. When considering a translational mode whose two rotor translational modal deflections are − 90° out-of-phase, the right-hand side of the first equality of Eq. (13) vanishes when calculated for the F_x and F_y components that are 90° out-of-phase. Thus, these components of F_x and F_y cannot excite these modes. They do, however, excite the translational modes where the x and y translations are 90° out-of-phase. Similarly, the F_x and F_y components that are − 90° out-of-phase cannot excite the translational modes whose two rotor translational deflections are 90° out-of-phase, but they do excite translational modes whose x and y translations are − 90° out-of-phase.

3.1. Reduction of the rotor translational vibration

As stated earlier, the rotor translational vibrations only involve translational modes and their associated modal coordinates in Eq. (13). From Eq. (13) and the known frequency content of F_x and F_y in Eqs. (4) and (5), the steady state oscillation frequencies of every $a_i(t)$ associated with a translational mode are $(jN \pm 1)\Omega$ for $j = 1, 2, \ldots, \infty$. Thus, the rotor steady state translational vibrations $x(t)$ and $y(t)$ have the same frequencies. The rotor has no rotation for translational modes, so μ is not considered here (but it is calculated later). Based on the properties of the translational modes as derived in [6,7] and the structure of F_x and F_y in Eqs. (4) and (5), the response components of $x(t)$ and $y(t)$ corresponding to frequency $(jN - 1)\Omega$ are 90° out-of-phase with equal amplitudes, whereas the response components corresponding to frequency $(jN + 1)\Omega$ are − 90° out-of-phase with equal amplitudes. Thus, we know that

$$x(t) = \sum_{j=1}^{\infty} \left\{ A_j \cos[(jN-1)\Omega t] + B_j \cos[(jN+1)\Omega t] \right\}, \tag{14a}$$

$$y(t) = \sum_{j=1}^{\infty} \left\{ A_j sin[(jN-1)\Omega t] - B_j sin[(jN+1)\Omega t] \right\}, \tag{14b}$$

where A_j and B_j are the constant amplitudes of each vibration harmonic.

The absorber response is decomposed as $s_g^{(i)}(t) = \hat{s}_g^{(i)}(t) + \tilde{s}_g^{(i)}(t)$, where $\hat{s}_g^{(i)}$ arises from excitation of the translational modes and $\tilde{s}_g^{(i)}$ arises from excitation of the rotational modes. Absorber modes are not excited by the substructure forces and do not contribute to the steady state response, as seen in Eq. (13). They play no role in this study, but they are important when one considers systems without perfect symmetry. From the considerations above, the absorber response $\hat{s}_g^{(i)}$ can be written as

$$\hat{s}_g^{(i)}(t) = \sum_{j=1}^{\infty}\left\{C_{g,j}^{(i)}\sin\left[(jN-1)\Omega t-\beta_g^{(i)}\right]+D_{g,j}^{(i)}\sin\left[(jN+1)\Omega t+\beta_g^{(i)}\right]\right\}, \tag{15}$$

where $C_{g,j}^{(i)}$ and $D_{g,j}^{(i)}$ denote steady state vibration amplitudes. The addition or subtraction of the spacing angle $\beta_g^{(i)}$ as a phase angle in Eq. (15) is necessary because of the phase angle properties of translational modes [6,7]. For these modes, which are the only ones contributing to $\hat{s}_g^{(i)}$, the phases of the absorber motions are exactly determined by the absorber spacing angles. The phase angles of the absorber coordinates of all the translational modes that are excited by the forces with frequency $(jN-1)\Omega$ and $(jN+1)\Omega$ are $\beta_g^{(i)}$ and $-\beta_g^{(i)}$, respectively. Hence, because of the known modal properties, the expression in Eq. (15) exactly depicts the phase angles of the steady state absorber vibrations for the two sets of excitation frequencies.

From Eq. (8), the equation of motion that governs the ith absorber of the gth group is

$$-\ddot{x}\,\sin\beta_g^{(i)}+\ddot{y}\cos\beta_g^{(i)}+\left(l_g+r_g\right)\ddot{\mu}+\ddot{s}_g^{(i)}+2\Omega\dot{x}\cos\beta_g^{(i)}+2\Omega\dot{y}\sin\beta_g^{(i)}+\Omega^2x\sin\beta_g^{(i)}-\Omega^2y\cos\beta_g^{(i)}+\Omega^2\frac{l_g}{r_g}s_g^{(i)}=0. \tag{16}$$

Substitution of the steady state vibration responses in Eqs. (14a), (14b) and (15) into Eq. (16) yields

$$\sum_{j=1}^{\infty}\left(\left\{(jN)^2A_j+\left[(jN-1)^2-n_g^2\right]C_{g,j}^{(i)}\right\}sin\left[(jN-1)\Omega t-\beta_g^{(i)}\right]\ +\left\{(jN)^2B_j-\left[(jN+1)^2-n_g^2\right]D_{g,j}^{(i)}\right\}sin\left[(jN+1)\Omega t+\beta_g^{(i)}\right]\right)=0, \tag{17}$$

where $n_g=\sqrt{l_g/r_g}$ is the tuning order of the gth group of absorbers. Multiplication of Eq. (17) by $sin[(mN-1)\Omega t+\beta_g^{(i)}]$ for integer m (and subsequently $sin[(mN+1)\Omega t-\beta_g^{(i)}]$) and integration over the interval $[0, 2\pi/\Omega]$ yields that all coefficients in Eq. (17) vanish. Therefore, the relations between the rotor translation and absorber motion amplitudes are

$$A_j=\frac{n_g^2-(jN-1)^2}{(jN)^2}C_{g,j}^{(i)}, \tag{18a}$$

$$B_j=-\frac{n_g^2-(jN+1)^2}{(jN)^2}D_{g,j}^{(i)}, \tag{18b}$$

$$i=1,2,\ldots,N_g,\qquad g=1,2,\ldots,p,\qquad j=1,2,\ldots,\infty. \tag{18c}$$

The substructure force excitation is not evident in Eqs. (18a), (18b) and (18c). The relation between the rotor responses of interest and this original excitation source can be derived by substitution of Eqs. (18a), (18b) and (18c) into the equations of motion that govern the rotor translation in [6,7]. Nevertheless, Eqs. (18a), (18b) and (18c) are the simplest form to derive the tuning order results of interest in this work.

For a chosen frequency component (or excitation order) defined by j in Eqs. (4) and (5), A_j vanishes if one group of absorbers is tuned to order $jN-1$ ($n_g=jN-1$). Similarly, the elimination of B_j requires another group to be tuned to order $jN+1$. Thus, two groups of absorbers are needed for optimal translational vibration reduction. With two groups of absorbers tuned this way, the rotor translational motion vanishes for the excitation order defined by j. This is independent of the magnitude of substructure forces (although physical constraints on the absorber size and number limit how much vibration they can absorb in practical systems).

Translational modes are the only modes relevant to translation of the rotor. They are also the only mode type that is excited by the net forces F_x and F_y from the substructures. In translational modes, the absorbers exert a net force on the rotor but not a net torque [6,7]. By properly tuning of two sets of absorbers to orders $jN\pm1$, the net substructure force excitation is directed to the absorber response $\hat{s}_g^{(i)}$ (t) associated with translational modes. Rotor forces at frequencies $(jN\pm1)\Omega$ are induced by this absorber response. These resulting rotor forces exactly counteract the order $jN\pm1$ components of F_x and F_y, thereby eliminating rotor translational vibration at these orders.

In a numerical study of absorbers on a Sikorsky UH-60 helicopter rotor [5], the translational rotor hub displacement is reduced by using absorbers whose orders are tuned to $N-1$ (or $N+1$) only. This choice of $j=1$ is because that is typically the dominant excitation order of vibration in most applications. The helicopter rotor translational vibration was reduced significantly. Only one group of absorbers was used in [5]. Better vibration reduction is possible with two groups of absorbers tuned to the orders $N\pm1$.

In systems whose dominant vibration excitation order is for $j\neq1$, absorbers tuned to the orders $N\pm1$ are ineffective because A_j and B_j in Eqs. (18a), (18b) and (18c) do not vanish. Instead the tuning would be to orders $jN\pm1$. The dominant excitation order of vibration in the system should be known to optimally choose the tuning orders.

3.2. Reduction of the rotor rotational vibration

Calculation of the rotor rotational vibration requires only the rotational modes. Because the state space modal analysis gives a trivial equation in Eq. (13) for the rigid body rotational mode, yet the rigid body mode response is important, we use a slightly different argument for rotor rotation (compared to rotor translation above) that does not use Eq. (13) directly. The key remains the modal properties, however.

The relevant response is the sum of all the rotational modes multiplied by time-dependent modal coordinates $a_i(t)$. From this and the known properties of rotational modes, the steady state response from the rotational modes must have the form

$$\tilde{\mathbf{q}} = \left(0, 0, \mu, \underbrace{\tilde{s}_1, \tilde{s}_1, \ldots, \tilde{s}_1}_{N_1}, \underbrace{\tilde{s}_2, \tilde{s}_2, \ldots, \tilde{s}_2}_{N_2}, \ldots, \underbrace{\tilde{s}_p, \tilde{s}_p, \ldots, \tilde{s}_p}_{N_p}\right)^T. \tag{19}$$

Substitution of Eq. (19) into the equations of motion in Eq. (8) yields

$$\left[J_r + \sum_{g=1}^{p} N_g m_g \left(l_g + r_g\right)^2\right]\ddot{\mu} + \sum_{g=1}^{p} N_g m_g \left(l_g + r_g\right)\ddot{\tilde{s}}_g = T, \tag{20a}$$

$$N_g m_g \left(l_g + r_g\right)\ddot{\mu} + N_g m_g \ddot{\tilde{s}}_g + \Omega^2 N_g m_g \frac{l_g}{r_g}\tilde{s}_g = 0, \qquad g = 1, 2, \ldots, p. \tag{20b}$$

The two rotor translation equations are not shown here because the forces F_x and F_y excite only the translational modes but not the rotational modes. From Eqs. (20a) and (20b) and the known frequency content of T in Eq. (7), the responses have the form

$$\mu(t) = \frac{l_g}{r_g}\left(l_g + r_g\right)G_1 t^2 + G_2 t + G_3 + \sum_{j=1}^{\infty} E_j \sin(jN\Omega t), \tag{21a}$$

$$\tilde{s}_g(t) = -G_1 + \sum_{j=1}^{\infty} H_{g,j} \sin(jN\Omega t), \qquad g = 1, 2, \ldots, p, \tag{21b}$$

$$G_1 = \frac{NLS_0 r_g \left(l_g + r_g\right)}{\left[J_r + \sum_{g=1}^{p} N_g m_g \left(l_g + r_g\right)^2\right] l_g \Omega^2}, \tag{21c}$$

where oscillation amplitudes E_j and $H_{g,j}$ are derived below. The constants G_2 and G_3 are, if desired, determined by the initial conditions for μ. We consider G_1 to be small here because we are discussing steady-state vibrations about a mean rotor speed.

Substitution of Eqs. (21a), (21b) and (21c) (with $\tilde{s}_g(t) = \tilde{s}_g^{(i)}(t)$) into Eq. (16) and recognizing that $x(t) = y(t) = 0$ for response in rotational modes give

$$\sum_{j=1}^{\infty}\left\{\left(l_g + r_g\right)(jN)^2 E_j + \left[(jN)^2 - n_g^2\right]H_{g,j}\right\} sin(jN\Omega t) = 0. \tag{22}$$

Satisfaction of Eq. (22) requires

$$E_j = \frac{n_g^2 - (jN)^2}{\left(l_g + r_g\right)(jN)^2} H_{g,j}. \tag{23}$$

When the absorbers are tuned to the jNth order ($n_g = jN$), E_j vanishes, and hence the rotor rotational vibration is eliminated at this excitation order. Unlike rotor translation, rotor rotation can be eliminated with a single group of absorbers. Note that the vibration can be reduced by tuning the absorbers close to order jN, and such detuning has some advantages when considering the effects of nonlinearity and mistuning among absorbers [15–17].

Taylor [8], Den Hartog [9], and Thomson [10] derived Eq. (23) and Newland [15] derived a nonlinear version of it. All of these works used a purely rotational degree of freedom model. The current derivation shows that Eq. (23) still holds for the more general model with rotor translation. The reason is the separation of modes into the three categories of translational, rotational, and absorber modes. Only rotational modes affect rotor rotation, and those modes are unchanged whether rotor translation is admitted or neglected. While all of this is true for small motions, that is, linear models, nonlinear effects can couple the responses in the different mode types.

The net torque T from the substructures excites only the rotational modes, and they are the only modes relevant to rotor rotation. The absorbers exert a net torque but no net force on the rotor for rotational modes [6,7]. Tuning a group of absorbers to order jN channels the net substructure torque excitation into absorber response $\tilde{s}_g^{(i)}(t)$ associated with rotational modes. This absorber response generates a rotor torque at frequency $jN\Omega$ that exactly counteracts the substructure torque at frequency $jN\Omega$.

4. Discussion

Eqs. (18a), (18b), (18c) and (23) reveal that the absorbers should be tuned to orders $jN \pm 1$ and jN to counteract the rotor translational and rotational vibrations, respectively. This requires three absorber groups for each set of substructure forces at a given excitation order defined by j. This is generally not practical, so the effects of non-optimal tuning must be considered. Eqs. (18a), (18b), (18c) and (23) indicate that the rotor translational and rotational amplitudes of each harmonic are proportional to the differences between the squares of the absorber tuning order and the desired excitation orders $jN \pm 1$ and jN, if the absorbers are not tuned properly. Therefore, any large differences between the absorber tuning order and another excitation order with non-trivial forcing excitation result in large rotor vibration amplitude in the other excitation order. This behavior is comparable to "spillover" in modal control [18]. In addressing one problem, the absorbers potentially create another problem. This is an inherent risk for systems with excitation at multiple orders, especially if the excitation orders differ widely in frequency. For example, in a system where two excitation orders defined by j_1 and j_2 are important yet these excitation orders have a large difference, tuning an absorber group to eliminate the vibration of either one of these two excitation orders causes a large vibration amplitude in the other order. Thus, two groups of absorbers that are tuned to these two excitation orders are needed in such systems to eliminate the vibrations of both orders. This spillover effect is similar to the result shown by Vidmar et al. [19] that different harmonics can "crosstalk". Therefore, some absorber groups are tuned to eliminate the vibration of the most troublesome excitation order, while the other groups could be tuned to other orders to avoid the spillover effect. Once any specific rotor vibration amplitude in Eqs. (18a), (18b), (18c) or (23) is eliminated by one group, however, it will not be adversely affected by the other attached groups.

Three groups of absorbers that are tuned to orders $jN \pm 1$ and jN are needed to eliminate the rotor translational and rotational vibrations at harmonic j as derived previously. In practical applications, however, the number of absorbers that can be attached is limited by space constraints. This restriction is a minor issue when the number of substructures or the dominant excitation order defined by j are large because the numerators in Eqs. (18a), (18b), (18c) and (23) can all be made simultaneously small (even if not zero) while the denominators are all large. To achieve this, a group of absorbers with a chosen tuning order between $jN \pm 1$ keeps the rotor translational and rotational amplitudes at harmonic j in Eqs. (18a), (18b), (18c) and (23) small, even though the vibration at the excitation order defined by j is not fully eliminated. Here we assumed that the j with maximum force excitation at frequencies $jN \pm 1$ in Eqs. (4) and (5) (large $P_{jN\pm1}$, $Q_{jN\pm1}$, $S_{jN\pm1}$, and $T_{jN\pm1}$) is the same j having maximum torque excitation at frequency jN in Eq. (7) (large P_{jN}, Q_{jN}, S_{jN}, and T_{jN}).

In a system with two substructures attached to the rotor ($N = 2$), the vibration order for $j = 1$ is eliminated when one of the two groups of absorbers is tuned to order $jN - 1 = 1$. This is the unity tuning group. One translational mode has zero rotor motion for unity tuning case [6,7]. The rotor forces from the absorbers are self-balanced in this translational mode. Thus, this mode is not effective in counteracting the rotor translational vibration. The rotor translational vibration is counteracted by the rotor forces from the absorbers in the other translational modes.

According to Eqs. (18a), (18b), (18c) and (23), the rotor vibration amplitudes can be reduced at any rotor speed by tuning the absorber properly to the orders specified above. With zero damping in the system and perfect tuning of the absorbers, the rotor vibrations in Eqs. (18a), (18b), (18c) and (23) at the chosen order are zero at any speed. Thus, the optimal tuning orders of the absorbers for rotor translational and rotational vibration reductions are independent on the system rotor speed.

More generally, the absorbers can be designed to counteract vibrations resulting from periodic external forces and torques not arising from cyclically symmetric substructures. These forces and torques should first be expressed in the rotating basis $\{\mathbf{e}_1^0, \mathbf{e}_2^0\}$ as done in Eqs. (1a), (1b), (2a), (2b), (3), (4), (5). The subsequent derivations of the optimal absorber tuning order are identical.

5. Conclusions

This paper derives the optimal tuning of CPVAs that are used to counteract the translational and rotational vibrations of a rotor with N cyclically symmetric substructures attached to it. The results derive general response features, and these show how one should tune the absorbers for optimal performance. The rotor translational and rotational vibrations are eliminated by groups of absorbers that are tuned to orders $jN \pm 1$ and jN, respectively. To optimally eliminate rotor translational vibration at a given excitation order, two groups of absorbers are required, whereas one group suffices to eliminate rotor rotation at a given excitation order. The highly structured vibration mode properties of CPVA systems are critical in the derivation. Vibration reduction at a desired excitation order using CPVAs may potentially increase the vibration at other orders. To address this potential problem, some absorber groups could be tuned to orders other than the excitation order of primary concern. Absorbers with order between $jN \pm 1$ are effective in reducing both the rotor translational and rotational vibrations at the excitation order defined by j for systems with a large number of substructures or with large j.

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