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Vibration reduction in a tilting rotor using centrifugal pendulum vibration absorbers

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ABSTRACT

This paper investigates vibration reduction in a rigid rotor with tilting, rotational, and translational motions using centrifugal pendulum vibration absorbers (CPVAs). A linearized vibration model is derived for the system consisting of the rotor and multiple sets of absorbers tuned to different orders. Each group of absorbers lies in a given plane perpendicular to the rotor rotation axis. Gyroscopic system modal analysis is applied to derive the steady-state response of the absorbers and the rotor to external, rotor-order, periodic forces and torques with frequency $m\Omega$, where Ω is the mean rotor speed and m is the engine order (rotor-order). It is found that an absorber group with tuning order m is effective at reducing the rotor translational, tilting, and rotational vibrations, provided certain conditions are met. When the periodic force and torque are caused by N substructures that are equally spaced around the rotor, the rotor translational and tilting vibrations at order j are addressed by two absorber groups with tuning orders $jN \pm 1$. In this case, the rotor rotational vibration at order j can be attenuated by an absorber group with tuning order jN. The results show how the response depends on the load amplitudes and order, the rotor speed, and design parameters associated with the sets of absorbers, most importantly, their tuning, mass, and plane of placement. In the ideal case with zero damping and exact tuning of the absorber sets, the vibrations can be eliminated for a range of loads over which the linearized model holds. The response for systems with detuned absorbers is also determined, which is relevant to applications where small detuning is employed due to robustness issues, and to allow for a larger range of operating loads over which the absorbers are effective. The system also exhibits undesirable resonances very close to these tuning conditions, an issue that is difficult to resolve and deserves further investigation.

1. Introduction

Centrifugal pendulum vibration absorbers (CPVAs) are order tuned absorbers whose frequencies automatically scale with the rotor speed. When properly tuned, they can reduce rotor rotational vibrations at a given order. The response of CPVA systems for rotational motions has been widely investigated, and their linear and nonlinear dynamics, including tuning

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strategies, are well understood; see, for example, [1–4] and the references cited therein. In terms of using CPVAs to address translational vibrations, Bramwell et al. [5] described how rotor translational vibration can be reduced by tuning the absorbers to orders $N \pm 1$ when N cyclically symmetric substructures are attached to the rotor. Bauchau et al. [6] numerically demonstrated translational vibration reduction in the four-bladed rotor of a Sikorsky UH-60 helicopter using absorbers tuned to the third or fifth order. Cronin [7] carried out a study of shaking vibration reduction in four-cylinder automotive engines. More recently, Shi et al. [8] analytically derived results showing how order jN translational vibration can be reduced using two groups of absorbers tuned to orders $jN \pm 1$, while another absorber group with tuning order jN is used reduce the rotational vibration at order jN. They provided expressions for the amplitudes and phases of the response of the absorbers and the rotor. The modal properties of CPVA systems with rotor translation and rotation derived by Shi and Parker [9,10] are used extensively in that derivation, and in the present work. These unique modal properties result from the cyclically symmetric arrangement of the CPVAs [11,12].

The present study extends previous results by accounting for tilting motions of the rotor and using CPVAs to address rotational, translational, and tilting vibrations. The model consists of a rigid rotor with rotational, tilting, and translational degrees of freedom supported by bearings of finite transverse stiffness and fitted with sets of CPVAs in a plane perpendicular to the rotor [13]. The goal is to investigate how one can tune and place sets of absorbers to reduce rotational, translational, and tilting vibrations of the rotor when it is subject to rotor-order forces and torques.

The response of CPVAs used for rotational vibration reduction, when lightly damped and moving with small amplitude, is proportional to the rotational excitation amplitude and the absorber detuning (specifically, the difference between the square of the excitation order and the square of the absorber tuning order), and inversely proportional to the absorber mass and Ω^2. If the absorber amplitudes exceed linear limits, nonlinear effects must be included [2,4,14,15]. Thus, these rotational vibration absorbers are most effective when they are tuned close to the excitation order, are lightly damped, and have sufficient mass (more generally, rotational inertia relative to the rotor central axis) to operate with sufficiently small absorber amplitudes. In this paper, we derive analogous results for the linearized response of a rotor with tilting, translational, and rotational degrees of freedom, to which are attached sets of absorbers tuned to different orders that address these vibrations. We first consider absorbers that address translational, tilting, and rotational vibrations of the rotor resulting from externally applied rotor-order forces and torques with frequency $m\Omega$. The system response calculated by gyroscopic modal analysis indicates that the rotor vibrations are reduced (eliminated, respectively) by an absorber group tuned close to (exactly at, respectively) order m. This absorber group is most effective when placed radially far from the rotor center of mass (COM) and located in, or close to, the same plane as that where the excitation is applied. The case of periodic forces and torques resulting from N substructures equally spaced around the rotor in a given plane is considered next. These loading conditions are derived in [8] and result in harmonics of orders jN for the rotational load and $jN \pm 1$ for translational and tilting loads. The tilting torque acts on the rotor when the plane with the excitation loads is offset from the rotor COM. Two absorber groups tuned close to (exactly at) orders $jN \pm 1$ reduce (eliminate) both the rotor translational and tilting vibrations caused by the net substructure force at order j, while another absorber group with tuning order close to (exactly at) jN reduces (eliminates) rotor rotational vibration excited by the substructures. Thus, the absorber tuning strategy for rotor tilting vibration is the same as that for rotor translational vibration [8], but the placement of absorbers along the rotor axis must also be considered.

In Section 2 we present the linearized system model and describe its modal properties, summarizing results from [9,10] as needed for the present analysis. In Sections 3 and 4 we consider the case of lateral forces and a rotor torque at a single order, deriving results for the general system response and describing how one achieves vibration reduction, or elimination in the ideal case, for all rotor degrees of freedom. This analysis relies heavily on the modal properties derived in [9,10]. Section 5 describes how one employs superposition to extend the results to the case of multiple harmonic loading that arises from substructures that are symmetrically placed about the rotor in a given plane, for example, a helicopter rotor.

2. Linear system model and modal analysis

The model (Fig. 1) consists of a rigid rotor with one rotational, two tilting, and two lateral translational degrees of freedom fitted with a total of N_a CPVAs, for a total of N_a+5 degrees of freedom. The rotor is supported by isotropic translational and rotational stiffnesses, while damping and gravity are ignored. The rotor spins with average speed Ω and is subjected to rotor-order loads at frequency $m\Omega$ (for integer m) consisting of external lateral forces applied in a plane that is offset from the rotor center of mass, denoted as COM, and a torque, also at frequency $m\Omega$, acting about the rotor axis. Different groups of absorbers are cyclically placed around the rotor, each group in a different plane that may be offset from both the rotor COM and the plane of loading. We begin by describing the external loads acting on the rotor, and then turn to the equations of motion.

We develop the model for the case where there are lateral forces at a single order m acting on the rotor in a given plane. Each absorber set lies in its own plane, and these are generally offset from the COM and from the plane with the loads. There is also a torque of order m acting about the rotor axis. As shown in Fig. 1, the oriented distance (that is, with a sign indicating direction) from the COM to the plane with the external loads is L, and the oriented distance from the COM to the plane with absorber group g is L_g. The fixed inertial frame is expressed by the basis with unit vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. The rotor-based frame

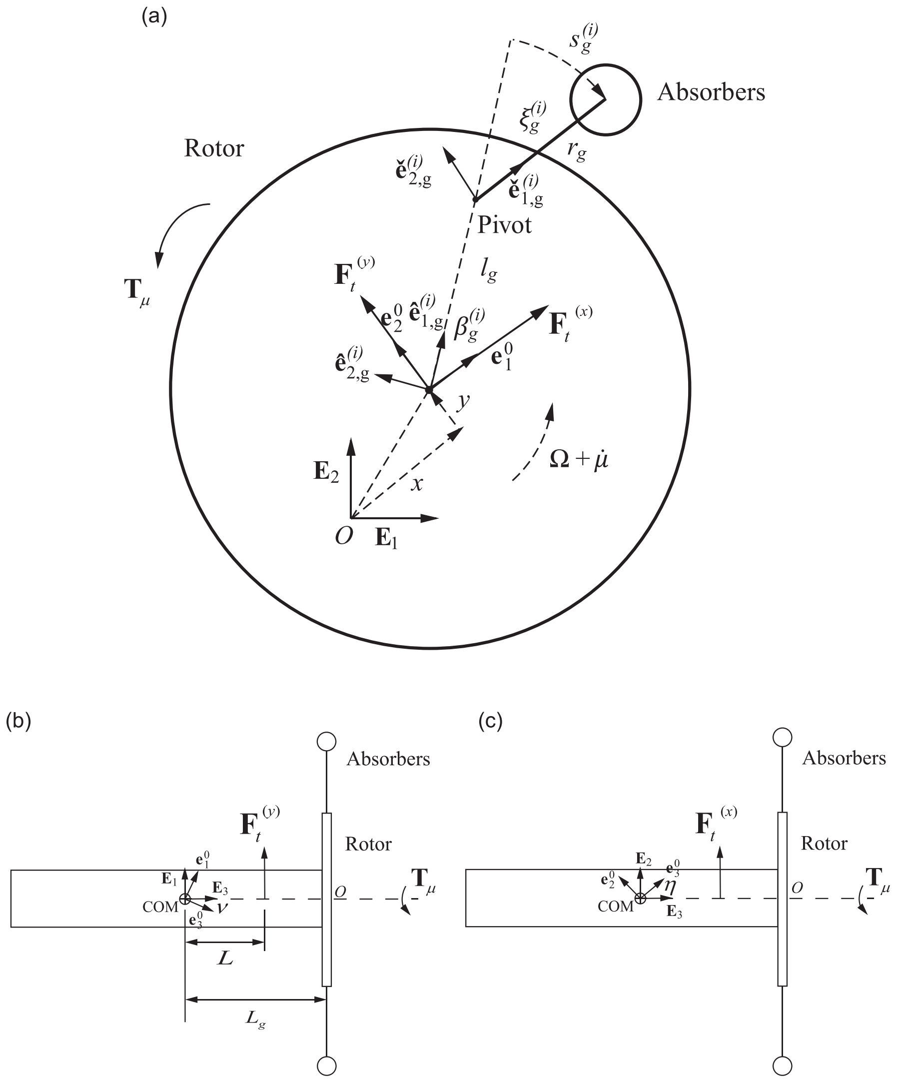

Fig. 1. Bases and coordinates used in the system consisting of a rotor with *p* groups of cyclically symmetric CPVAs attached to it. (a) Side view of the rotor in its deflected position. (b) Front view of the system in its undeflected configuration. (c) Top view of the system in its undeflected configuration.

is described by unit vectors $\{\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0\}$ that rotate at a constant speed Ω about $\mathbf{e}_3^0$. Fig. 1(a) shows these bases. The rotor in-plane translations along the $\mathbf{e}_1^0$ and $\mathbf{e}_2^0$ directions are denoted by x and y, respectively.

As shown in Fig. 2, the rotor-order force $\mathbf{F}_t$ with constant amplitude is applied on the rotor along $\hat{\mathbf{e}}_1^0$, where the basis $\left\{\hat{\mathbf{e}}_1^0, \hat{\mathbf{e}}_2^0, \hat{\mathbf{e}}_3^0\right\}$ is rotating at a constant speed $m\Omega$ relative to the rotor-fixed basis $\{\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0\}$. Thus, $\mathbf{F}_t$ can be expressed in $\{\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0\}$ basis as shown in Eq. (1a), where the x and y components of the rotor-order applied force $\mathbf{F}_t$ have equal amplitudes and are phase shifted by 90° relative to one another. The line of action of $\mathbf{F}_t$ passes through the rotor axis of rotation. It does not contribute to the rotor torque but does produce tilting moments about the COM. The lateral force and torque can be expressed as

$$\mathbf{F}_t = F\hat{\mathbf{e}}_1^0 = F_x\mathbf{e}_1^0 + F_y\mathbf{e}_2^0$$
$$= F\left[\cos(m\Omega t)\mathbf{e}_1^0 + \sin(m\Omega t)\mathbf{e}_2^0\right], \tag{1a}$$

$$\mathbf{T}_\mu = T_\mu\mathbf{e}_3^0 = T\cos(m\Omega t + \phi)\mathbf{e}_3^0, \tag{1b}$$

Because the force acts at a distance L from the COM, it exerts a tilting torque about the COM according to

$$\mathbf{T}_t = L\mathbf{e}_3^0 \times \mathbf{F}_t = -LF_y\mathbf{e}_1^0 + LF_x\mathbf{e}_2^0. \tag{2}$$

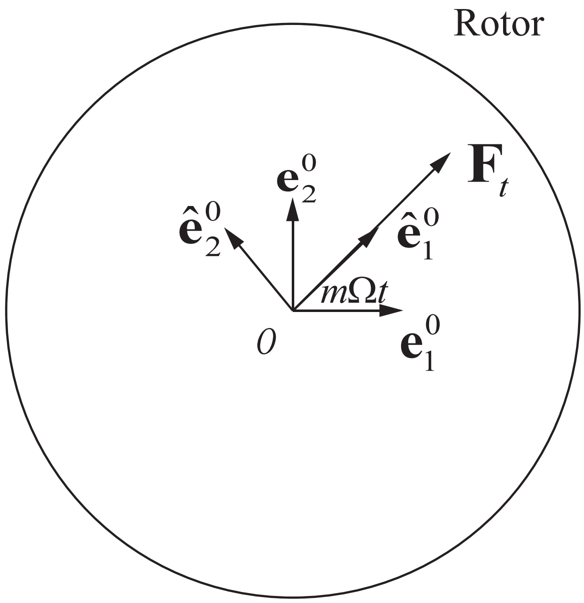

Fig. 2. The lateral rotor-order force $\mathbf{F}_t$ applied on the rotor.

The rotor rotational vibration about its axis is given by the angular deviation μ away from the nominal rotor angle at constant speed Ω, such that the rotor speed is expressed as $\Omega+\dot{\mu}$ where $\Omega \gg |\dot{\mu}|$ for small amplitude rotational vibrations.

The model has p groups of CPVAs, assumed to have bifilar suspension so that they act as point masses [16–18], each moving along a fixed path on the rotor, shown for the case of a circular path in Fig. 1. Absorber group g possesses N_g identical absorbers placed with cyclic symmetry around the rotor in a given plane a distance L_g from the rotor COM. The total number of absorbers is $N_a = \sum_{g=1}^{p} N_g$. The rotor-fixed basis $\left\{\hat{\mathbf{e}}_{1,g}^{(i)}, \hat{\mathbf{e}}_{2,g}^{(i)}, \hat{\mathbf{e}}_{3,g}^{(i)}\right\}$ is defined such that $\hat{\mathbf{e}}_{1,g}^{(i)}$ points from the rotor center to the pivot point of the ith absorber in the gth group. The absorber pivot position is described by the fixed angle $\beta_g^{(i)}$ between $\mathbf{e}_1^0$ and $\hat{\mathbf{e}}_{1,g}^{(i)}$. The basis $\left\{\check{\mathbf{e}}_{1,g}^{(i)}, \check{\mathbf{e}}_{2,g}^{(i)}, \check{\mathbf{e}}_{3,g}^{(i)}\right\}$ is fixed on the ith absorber in the gth group. The position of each absorber during its motion is described by an arc-length coordinate along its path denoted by $s_g^{(i)}$. The radial distance between the rotor axis and the pivot position of each absorber in the gth group is l_g, and the circular path radius of each absorber in the gth group is r_g. Thus, the tuning order for each absorber in the gth group is $n_g = \sqrt{l_g/r_g}$ [1,16].

Fig. 1(b) and (c) shows the undeflected rotor viewed from directions normal to the rotor axis. The two rotor tilting degrees of freedom ν and η are shown in Fig. 1(b) and (c), respectively. The distance L_g from the COM to the plane containing absorber group g is shown in Fig. 1(b) and (c) (note that L_g is denoted as L in [13]).

The rotor mass, moment of inertia about its rotation axis, and tilting moment of inertia about any axis through the rotor COM and perpendicular to the rotation axis are m_r, J_r, and J_t, respectively. The isotropic translational rotor bearing stiffness is k_r, and the isotropic rotational tilting stiffness is K_t. Each absorber in the gth group has mass m_g and, as noted above, is tuned to order $n_g = \sqrt{l_g/r_g}$. For bifilar absorbers one can include the moment of inertia of each absorber about its COM in the effective rotor inertia J_r and treat the absorbers as point masses, because the absorbers rotate with the rotor. The moments of inertia of the suspension rollers are neglected here but can be included in the tuning calculations [16,19].

The linearized equations of motion of this three-dimensional system were derived in [13] and have the form

$$\mathbf{M}\ddot{\mathbf{q}} + \Omega\mathbf{G}\dot{\mathbf{q}} + (\mathbf{K}_b - \Omega^2\mathbf{K}_\Omega)\mathbf{q} = \mathbf{F}, \tag{3a}$$

$$\mathbf{q} = (\mathbf{q}_r, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_p)^T, \tag{3b}$$

with the rotor coordinates given by

$$\mathbf{q}_r = (x, y, \mu, \nu, \eta)^T, \tag{4}$$

and the absorber coordinates expressed as

$$\mathbf{q}_g = \left(s_g^{(1)}, s_g^{(2)}, \ldots, s_g^{(N_g)}\right)^T, \quad g = 1, 2, \ldots, p. \tag{5}$$

The forcing vector from Eqs. (1a) and (1b) is

$$\mathbf{F} = (F_x, F_y, T_\mu, -LF_y, LF_x, \underbrace{0, 0, \ldots, 0}_{N_a})^T. \tag{6}$$

Details of these equations are given in [13]. In this work we subsequently present more specific forms of these equations as needed for the present analysis. The eigenvalue problem associated with Eq. (3a) is

$$\lambda^2 \mathbf{M}\boldsymbol{\phi} + \lambda\Omega\mathbf{G}\boldsymbol{\phi} + (\mathbf{K}_b - \Omega^2\mathbf{K}_\Omega)\boldsymbol{\phi} = \mathbf{0}. \tag{7}$$

Eq. (3a) has the standard form for a gyroscopic system. These systems are commonly solved by a state space method in the form [20–22]

$$\mathbf{A}\dot{\mathbf{z}} + \mathbf{B}\mathbf{z} = \mathbf{g}, \quad \mathbf{z} = (\dot{\mathbf{q}} \quad \mathbf{q})^T, \quad \mathbf{g} = (\mathbf{F} \quad \mathbf{0}_{1\times(N_a+5)})^T, \tag{8a}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{M} & \mathbf{0}_{(N_a+5)\times(N_a+5)} \\ \text{symmetric} & \mathbf{K}_b - \Omega^2\mathbf{K}_\Omega \end{pmatrix}, \tag{8b}$$

$$\mathbf{B} = \begin{pmatrix} \Omega\mathbf{G} & \mathbf{K}_b - \Omega^2\mathbf{K}_\Omega \\ \text{skew}-\text{symmetric} & \mathbf{0}_{(N_a+5)\times(N_a+5)} \end{pmatrix}. \tag{8c}$$

The associated eigenvalue problem is $\lambda\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u} = \mathbf{0}$, where $\mathbf{u} = (\lambda\boldsymbol{\phi}, \boldsymbol{\phi})^T$ is the eigenvector associated with the eigenvalue λ and $\boldsymbol{\phi}$ is a complex-valued eigenvector of Eq. (7). The eigenvectors satisfy the orthogonality relations $\overline{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_j = \overline{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_j = 0$ for $i \neq j$, where an overbar represents the complex conjugate. These eigenvectors are normalized such that $\overline{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i = 1$, in which case, $\overline{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_i = -\lambda_i$. For the rigid-body rotational mode, it is clear that $\overline{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i = \overline{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_i = 0$; this mode is only relevant to the rotor rotational dynamics, as discussed later, and it requires a different treatment from the other modes [8].

This system possesses three distinct mode types: rotational, translational-tilting, and absorber modes [13]. These modal properties are critical to the analysis that follows. The rotational modes have only rotor rotation, with no rotor translation or tilting. The absorbers within each group move identically and in unison. Rotational modes are the relevant modes to analyze CPVA systems with rotors having only rotational vibration. There exist two such modes when a single absorber group is applied. One of these two modes is a rigid body mode with zero frequency, and the eigenfrequency of the other rotational mode is

$$\omega = \Omega n_g\sqrt{1 + \frac{N_g m_g (l_g + r_g)^2}{J_r}}, \tag{9}$$

from which it is seen that the resonance is close to the desired tuning Ωn_g when the inertia of the absorbers is small compared to that of the rotor. Each additional absorber group results in an additional rotational mode, whose natural frequency is close to the desired tuning of the additional absorber group [10]. The addition of an absorber group slightly alters the prior natural frequencies and vibration modes.

The translational-tilting modes involve coupled rotor translations and tilting motions with no rotor rotation. The absorbers within one group vibrate with $2\pi/N_g$ phase difference between the neighboring absorbers. There exist six such modes and their frequencies are distinct for systems with a single absorber group. Each additional absorber group adds two mode translational-tilting modes [10].

The absorber modes are associated with absorber motions but no rotor motions of any kind because the absorber motions are such that their net effect on the rotor is zero. Each absorber group with N_g absorbers has $N_g - 3$ degenerate absorber modes, and the repeated natural frequency of these modes for a particular group is Ωn_g [10]. Since these modes are degenerate, there is some freedom in choosing the mode shapes, which is done in the manner described in [13].

The forces acting on the rotor excite only the translation-tilting modes of vibration, through $\mathbf{F}_t$ and $\mathbf{T}_t$, while the torque $\mathbf{T}_\mu$ excites only the rotational modes. None of these loads excites the absorber modes. These facts greatly simplify the analysis. The mathematical justification for them is evident from the modal analysis solution to Eq. (8), as follows. We express the response in the form

$$\mathbf{z}(t) = \sum_{m=1}^{N_a+5} [a_m(t)\mathbf{u}_m + \overline{a}_m(t)\overline{\mathbf{u}}_m], \tag{10}$$

where the $a_m(t)$ are complex-valued modal coordinates and the $\mathbf{u}_m$ are the vibration modes. Substitution of this $\mathbf{z}(t)$ into Eq. (8), pre-multiplication by $\overline{\mathbf{u}}_m^T$, and use of the orthogonality relations yields decoupled equations for the modal coordinates, with the form for each equation depending on the mode type associated with each modal coordinate, specifically,

$$\dot{a}_m - \lambda_m a_m = \overline{\mathbf{u}}_m^T\mathbf{g}$$
$$= \begin{cases} \overline{\lambda}_m(\overline{x}_m F_x + \overline{y}_m F_y - \overline{\nu}_m L F_y + \overline{\eta}_m L F_x); & \text{if } m \text{ is for a translational}-\text{tilting mode} \\ \overline{\lambda}_m\overline{\mu}_m T; & \text{if } m \text{ is for a rotational mode} \\ 0; & \text{if } m \text{ is for an absorber mode} \end{cases}, \tag{11}$$

where x_m, y_m, μ_m, ν_m, and η_m are the complex-valued translational, rotational, and tilting motions of the rotor in the mode $\mathbf{u}_m$, respectively.

The rotor translational and tilting responses $(x(t), y(t), \nu(t), \eta(t))$ are affected only by the translational-tilting modes, so only these modes are considered in the investigation of translational and tilting vibration. Similarly, only the rotational modes can affect the rotor rotational response $\mu(t)$, and thus they are the only mode type considered for the investigation of the rotational vibration. These two types of responses, translation-tilting and rotational, can therefore be considered individually as done below.

3. Translational and tilting vibrations

The steady-state solution of the linear differential equation (11) is the periodic particular solution at the frequency $m\Omega$ for the periodic force in (1a). For a translational-tilting mode, the modal deflections x_m and y_m have equal amplitudes, and y_m is 90° out-of-phase with respect to x_m [13]. Given these relationships, and the previously discussed phase relationships for F_x and F_y, the responses of $x(t)$ and $y(t)$ have equal amplitudes and are 90° phase-shifted relative to each other. These amplitude and phase relations also hold for the tilting response components $\nu(t)$ and $\eta(t)$. The translation and tilting motions of the translational-tilting mode discussed in [13] also are such that $x(t)$ and $\nu(t)$ are $-90°$ out-of-phase. Therefore, the rotor translational and tilting responses have the form

$$x(t) = A\cos(m\Omega t), \tag{12a}$$

$$y(t) = A\sin(m\Omega t), \tag{12b}$$

$$\nu(t) = -C\sin(m\Omega t), \tag{12c}$$

$$\eta(t) = C\cos(m\Omega t), \tag{12d}$$

where the steady-state amplitudes A and C have the same sign, as required by the phase relationships between the rotor translation and tilting in translational-tilting modes [13]. The absorber responses have the form

$$\hat{s}_g^{(i)}(t) = E_g \sin(m\Omega t - \beta_g^{(i)}), \quad i = 1, 2, \ldots, N_g, \quad g = 1, 2, \ldots, p, \tag{13}$$

where E_g is the steady-state amplitude of each absorber in the gth group, which are equal under this form of excitation based on the system response calculated in Eqs. (10) and (11) and the modal properties of translational-tilting modes derived in [13]. The phase shifts $\beta_g^{(i)}$ arise from the phase relationships between the absorber degrees of freedom in the translational-tilting modes [13].

Having used modal analysis of Eq. (8) to decouple the modal coordinates, to separate the analysis of translation and tilting from rotation, and to establish the solution forms in Eqs. (12) and (13), we now turn to the direct equations of motion to determine A, C, and E_g. The linearized equations that govern the absorber motions in the gth group are [13]

$$\begin{aligned} &-\ddot{x}\sin\beta_g^{(i)} + \ddot{y}\cos\beta_g^{(i)} + \ddot{\mu}(l_g + r_g) - \ddot{\nu}L_g\cos\beta_g^{(i)} - \ddot{\eta}L_g\sin\beta_g^{(i)} + \ddot{s}_g^{(i)} \\ &+ 2\Omega\dot{x}\cos\beta_g^{(i)} + 2\Omega\dot{y}\sin\beta_g^{(i)} - 2\Omega\dot{\nu}L_g\sin\beta_g^{(i)} + 2\Omega\dot{\eta}L_g\cos\beta_g^{(i)} \\ &+ \Omega^2 x\sin\beta_g^{(i)} - \Omega^2 y\cos\beta_g^{(i)} + \Omega^2\nu L_g\cos\beta_g^{(i)} + \Omega^2\eta L_g\sin\beta_g^{(i)} + \Omega^2 s_g^{(i)}\frac{l_g}{r_g} = 0, \\ &i = 1, 2, \ldots, N_g, \quad g = 1, 2, \ldots, p. \end{aligned} \tag{14}$$

Substitution of Eqs. (12) and (13) into Eq. (14) yields

$$\left[(m+1)^2(A + L_g C) + (m^2 - n_g^2)E_g\right]\sin(m\Omega t - \beta_g^{(i)}) = 0, \quad g = 1, 2, \ldots, p, \tag{15}$$

which requires that

$$A + L_g C = \frac{n_g^2 - m^2}{(m+1)^2}E_g, \quad g = 1, 2, \ldots, p. \tag{16}$$

This relates the amplitudes of rotor translation A, rotor tilting C, and absorber motions E_g for the gth absorber group.

Substitution of Eqs. (12), (13), and (16) into the equations of motion in Eq. (3a) that govern the rotor translation and tilting motions (see [13]) yields the following general relationships between the rotor translational and tilting amplitudes

$$W_1 A + W_2 C = F/\Omega^2, \tag{17a}$$

$$W_2 A + W_3 C = LF/\Omega^2, \tag{17b}$$

$$W_1 = k_r/\Omega^2 - (m+1)^2\left(m_r + \sum_{g=1}^{p} N_g m_g\right) - \frac{(m+1)^4}{2}\sum_{g=1}^{p}\frac{1}{n_g^2 - m^2}N_g m_g, \tag{17c}$$

$$W_2 = -(m+1)^2 L_g \sum_{g=1}^{p}\left[1-\frac{(m+1)^2}{2(n_g^2-m^2)}\right] N_g m_g, \tag{17d}$$

$$W_3 = K_t/\Omega^2 - m^2\left[J_t + \sum_{g=1}^{p} N_g m_g \frac{(l_g+r_g)^2}{2}\right] + \sum_{g=1}^{p} N_g m_g \frac{(l_g+r_g)^2}{2} + W_2 L_g, \tag{17e}$$

Solving these for the rotor translational and tilting amplitudes yields

$$A = \frac{(W_2 L - W_3)F}{\Omega^2(W_2^2 - W_1 W_3)}, \tag{18a}$$

$$C = \frac{(W_2 - W_1 L)F}{\Omega^2(W_2^2 - W_1 W_3)}. \tag{18b}$$

The amplitude E_g of the gth absorber group is then determined by substitution of Eq. (18) into Eq. (16). Eq. (16) shows that the amplitude E_g of the gth absorber group depends explicitly only on the amplitudes of rotor translation and tilting and the properties n_g and L_g of the absorber group with no explicit dependence on the amplitudes or parameters of other absorber groups. In contrast, rotor translational and tilting amplitude depends on all absorber groups as seen in Eq. (18) and the summations in Eqs. (17c), (17d), and (17e).

Although perfect tuning $n_\gamma = m$ eliminates rotor translation and tilting in principal (as shown below in Eq. (19)), in practical applications the absorbers are tuned to an order close to m. The use of small detuning extends the operating amplitude of the absorbers by keeping them in the linear range over a larger range of loads, albeit at the expense of less vibration reduction [2,15,23]. When the γth group is tuned to order $n_\gamma = m - \epsilon$, where ϵ is the small detuning, and other absorber groups are tuned to other orders different from m, the amplitudes of the linear model rotor translational and tilting responses to the periodic force are, to leading order in ϵ, given by

$$A = -2m\epsilon \frac{F(L-L_\gamma)}{\Omega^2(m+1)^4 L_\gamma N_\gamma m_\gamma} \prod_{g\neq\gamma}\left(n_g^2 - m^2\right) + \mathcal{O}(\epsilon^2), \tag{19a}$$

$$C = -2m\epsilon \frac{F(L+L_\gamma)}{\Omega^2(m+1)^4 L_\gamma^2 N_\gamma m_\gamma} \prod_{g\neq\gamma}\left(n_g^2 - m^2\right) + \mathcal{O}(\epsilon^2). \tag{19b}$$

Eq. (19) indicates that the rotor translational and tilting amplitudes are proportional to the detuning. Thus, absorber groups with small detuning still reduce the rotor translation and tilting motions. This result shows that by tuning one set of absorbers exactly, that is, taking $n_\gamma = m$, or, equivalently, $\epsilon = 0$, the absorbers will completely eliminate rotor translational and tilting vibrations (in this idealized model without damping).

Eq. (19) highlights a potentially damaging effect of additional absorber groups tuned to orders away from m. If the tuning order n_g for one or more of these groups is well away from m, then one or more factors in Eq. (19) can be large, which increases rotor translation and tilting at order m. In theory, this effect is eliminated with perfect tuning ($n_\gamma = m$ giving $\epsilon = 0$) of the absorber group tuned to order m.

Substitution of Eq. (18) into Eq. (16), with one absorber group tuned such that $n_\gamma = m - \epsilon$, yields

$$E_\gamma = \lim_{n_\gamma \to m}(A + L_\gamma C)\frac{(m+1)^2}{n_\gamma^2 - m^2}$$

$$= \frac{2FL}{\Omega^2(m+1)^2 L_\gamma N_\gamma m_\gamma} \prod_{g\neq\gamma}\left(n_g^2 - m^2\right) + \frac{\partial E_\gamma}{\partial \epsilon}\Big|_{\epsilon=0}\epsilon + \mathcal{O}(\epsilon^2). \tag{20}$$

This expression is valid up to order ϵ for the case of small detuning. It indicates that the absorber amplitudes are proportional to the applied moment FL about the COM, and inversely proportional to the effective moment $N_\gamma m_\gamma L_\gamma \Omega^2$ produced by the acceleration of the total absorber mass $N_\gamma m_\gamma$ of absorber group γ. For these results to be valid, the absorbers must be designed to maintain small amplitudes for the given range of loading conditions, thus staying in the linear response regime.

In the typical design process, one knows the maximum effective torque, that is, the load-speed combination FL/Ω^2 (with units of moment of inertia), to be encountered. The absorber location L_γ and total mass $N_\gamma m_\gamma L_\gamma$ are chosen to achieve the maximum absorber amplitude E_γ allowed by hardware. In practice, the absorber parameters are often constrained by space and/or mass limitations, in which case perfect tuning does not allow the system to maintain linearity over the desired torque range. In this case, one can detune the absorbers, which sacrifices absorber effectiveness (by increasing amplitudes A, C), but gives a larger torque range over which the absorber amplitudes E_g remain in the linear range.

4. Rotational vibrations

Due to the modal properties of the system, only the rotational modes need to be considered in the calculation of rotor rotational response. As seen in Eq. (11), these modes are excited only by the external torque $\mathbf{T}_t$. Thus, rotor rotation analysis is complementary to the above analysis of translational-tilting response.

The steady-state rotational response is the sum of all the rotational modes multiplied by associated time-dependent modal coordinates $a_i(t)$. Based on this and the known modal properties of the rotational modes [13], the steady-state response associated with the rotor rotational vibration has the form

$$\tilde{\mathbf{q}} = \left(0, 0, \mu, 0, 0, \underbrace{\tilde{s}_1, \tilde{s}_1, \ldots, \tilde{s}_1}_{N_1}, \underbrace{\tilde{s}_2, \tilde{s}_2, \ldots, \tilde{s}_2}_{N_2}, \ldots, \underbrace{\tilde{s}_p, \tilde{s}_p, \ldots, \tilde{s}_p}_{N_p}\right)^T. \tag{21}$$

Substitution of this form into the equations of motion in Eq. (3a) (see [13]) yields a set of reduced equations for the rotational and absorber responses given by

$$\left[J_r + \sum_{g=1}^{p} N_g m_g (l_g + r_g)^2\right]\ddot{\mu} + \sum_{g=1}^{p} N_g m_g (l_g + r_g)\ddot{\tilde{s}}_g = T\cos m\Omega t, \tag{22a}$$

$$N_g m_g (l_g + r_g)\ddot{\mu} + N_g m_g \ddot{\tilde{s}}_g + \Omega^2 N_g m_g \frac{l_g}{r_g}\tilde{s}_g = 0, \quad g = 1, 2, \ldots, p. \tag{22b}$$

Recall that the rotor translational and tilting components of the equations of motion are not required here since they are not excited by $\mathbf{T}_t$.

Based on Eq. (22), the periodic components of the steady-state rotor rotation and absorber responses have the form,

$$\mu(t) = M\cos(m\Omega t), \tag{23a}$$

$$\tilde{s}_g(t) = V_g \cos(m\Omega t), \quad g = 1, 2, \ldots, p. \tag{23b}$$

In determining μ we consider only its oscillatory (zero-mean) component and ignore the arbitrary rotor angle set by initial conditions. In addition, because Ω captures the mean rotor speed, the average of $\dot{\mu}$ vanishes. These considerations eliminate the rigid body mode from the analysis.

Substitution of Eq. (23) into Eq. (14), along with the previously discussed stipulation that the rotor translations and tilting motions vanish when considering response of the rotational modes, yields the steady-state condition for rotor rotation and absorbers

$$\left[(l_g + r_g)m^2 M + (m^2 - n_g^2)V_g\right]\cos(m\Omega t) = 0, \quad g = 1, 2, \ldots, p, \tag{24}$$

which requires

$$M = \frac{n_g^2 - m^2}{(l_g + r_g)m^2}V_g, \quad g = 1, 2, \ldots, p. \tag{25}$$

This relates the rotational response of the rotor and that of the absorbers within each group, analogous to Eq. (16) for the case of the rotor translation-tilting response. Again, it indicates that one can eliminate rotor rotation by perfect tuning, $n_\gamma = m$, as is well known [1,2,24,25], but in the following we present a general analysis that is more useful for design purposes. As for rotor translation and tilting reduction, if one absorber group is tuned perfectly such that $n_\gamma = m$ then one can add absorber groups to eliminate rotor rotation at orders different from m without affecting the elimination of rotor rotation at order m by group γ.

Substitution of Eqs. (23) and (25) into the reduced rotor rotation equation of motion, Eq. (22a), gives the rotor rotational amplitude as

$$M = -\frac{T}{\Omega^2 W_4}, \tag{26a}$$

$$W_4 = \left[J_r + \sum_{g=1}^{p} N_g m_g (l_g + r_g)^2 \frac{n_g^2}{n_g^2 - m^2}\right]m^2. \tag{26b}$$

Note that W_4 is the effective rotational inertia of the system, which is infinite in the perfect tuning case, thus eliminating the rotational vibration of the rotor. Substitution of Eq. (26a) into Eq. (25) yields the amplitude of the gth absorber group as

$$V_g = -\frac{(l_g + r_g)m^2 T}{\Omega^2 (n_g^2 - m^2)W_4}, \quad g = 1, 2, \ldots, p. \tag{27}$$

One must account for the fact that in these forms for M and V_g, the factors $(n_g^2-m^2)$ appear in the numerators and denominators of different terms. For example, if one set of absorbers is perfectly tuned such that $n_\gamma = m$, then W_4 becomes infinite, rendering $M=0$. In this case, the product $(n_\gamma^2-m^2)W_4$ appearing in V_γ must be treated systematically and results in a finite absorber amplitude, as shown below.

For the case where one absorber group is tuned close to m, according to $n_\gamma = m-\epsilon$, the rotor rotational amplitude is approximated to leading order in ϵ by

$$M = 2\epsilon \frac{T}{\Omega^2 m N_\gamma m_\gamma (l_\gamma + r_\gamma)^2 n_\gamma^2} \prod_{g \neq \gamma} \left(n_g^2 - m^2\right) + \mathcal{O}(\epsilon^2). \tag{28}$$

As expected, this amplitude converges to zero as $\epsilon \to 0$. Eq. (28) shows that for a given level of detuning ($\epsilon \neq 0$) the absorbers are more effective with larger mass $N_\gamma m_\gamma$, larger radial distance from the rotor axis $(l_\gamma + r_\gamma)$, and at larger rotor speeds Ω, because these affect the torque that the absorbers can exert on the rotor. Like for rotor translation and tilting, other absorber groups tuned to orders other than m increase the rotor rotation amplitude when the γth group is detuned ($\epsilon \neq 0$).

The corresponding absorber amplitude for the group with detuned absorbers is

$$\begin{aligned} V_\gamma &= \lim_{n_\gamma \to m} \left[-\frac{(l_g + r_g) m^2 M}{(n_g^2 - m^2)} \right] \\ &= -\frac{T}{\Omega^2 N_\gamma m_\gamma (l_\gamma + r_\gamma) n_\gamma^2} \prod_{g \neq \gamma} \left(n_g^2 - m^2\right) + \frac{\partial V_\gamma}{\partial \epsilon}\Big|_{\epsilon = 0} \epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{29}$$

Note that Eq. (29) also applies for $\epsilon = 0$ (i.e., perfect tuning); it is the limit of Eq. (27) as $n_\gamma \to m$.

The design process for rotational absorbers follows similar to the translation-tilting case. Specifically, given a maximum value of effective torque T/Ω^2 that will be experienced, one designs an absorber group with $N_\gamma m_\gamma (l_\gamma + r_\gamma)$ sufficiently large to keep the absorber amplitude V_g in the linear range and within hardware limits on absorber motion amplitude. If this cannot be achieved with perfect tuning, one must detune (which generally means slightly overtune) the absorbers to keep V_g in the linear range and then determine if the associated reduction in rotor rotational vibration M from Eq. (28) is sufficient.

The above tuning considerations ignore the presence of resonances in the system. For small absorber masses (inertias), resonances can be close to the ideal tuning conditions. Thus, for robustness one must determine the frequencies (or orders) of these resonances and their proximity to the tuning conditions. One can then adjust the absorber tuning to avoid resonance based on the level of system uncertainties such as tolerances and nonlinear effects.

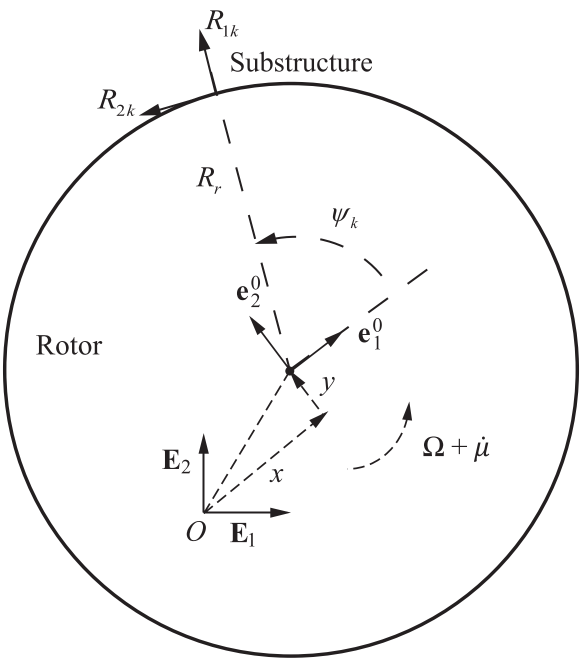

Fig. 3. The system containing a rotor with N cyclically symmetric substructures attached to it.

5. Reduction of vibrations caused by cyclically symmetric substructures

Here we consider the situation in which the periodic force and the rotational torque are caused by N cyclically symmetric substructures, as shown in Fig. 3. This leads to applied forces and torques with multiple harmonics. The use of superposition extends the results described above to this case.

We consider periodic forces with fundamental frequency Ω exerted by the kth substructure on the rotor, with the radial and tangential direction forces denoted by R_{1k} and R_{2k}. These forces are a distance R_r away from the axis of rotation. The net x- and y- forces resulting from these substructures are calculated by summing R_{1k} and R_{2k} along the $\mathbf{e}_1^0$ and $\mathbf{e}_2^0$ directions, which yields [8]

$$F_x = \frac{N}{2}\sum_{j=1}^{\infty}\{(P_{jN-1}+T_{jN-1})\cos[(jN-1)\Omega t]+(Q_{jN-1}-S_{jN-1})\sin[(jN-1)\Omega t] + (P_{jN+1}-T_{jN+1})\cos[(jN+1)\Omega t]+(Q_{jN+1}+S_{jN+1})\sin[(jN+1)\Omega t]\}, \tag{30a}$$

$$F_y = \frac{N}{2}\sum_{j=1}^{\infty}\{(P_{jN-1}+T_{jN-1})\sin[(jN-1)\Omega t]-(Q_{jN-1}-S_{jN-1})\cos[(jN-1)\Omega t] + (Q_{jN+1}+S_{jN+1})\cos[(jN+1)\Omega t]-(P_{jN+1}-T_{jN+1})\sin[(jN+1)\Omega t]\}. \tag{30b}$$

The coefficients P_i, Q_i, S_i, and T_i are computed from the Fourier series for R_{1k} and R_{2k} with a sign difference from [8] because of the definition of R_{2k} in Fig. 3. Note that the periodic forces F_x and F_y have equal amplitudes and frequency components at $(jN-1)\Omega$ and $(jN+1)\Omega$, $j=1,2,\ldots,\infty$, generated by the order N forces applied in the rotating frame. The $(jN-1)\Omega$ components of F_y are 90° phase-shifted relative to the corresponding terms in F_x, and similarly the $(jN+1)\Omega$ components of F_y are $-90°$ phase-shifted relative to the corresponding terms in F_x. The net rotational torque that the substructures exert on the rotor is calculated by summing the tangential forces R_{2k} multiplied by the moment arm R_r, resulting in [8]

$$T_\mu = -NR_r\left\{S_0+\sum_{j=1}^{\infty}[S_{jN}\cos(jN\Omega t)+T_{jN}\sin(jN\Omega t)]\right\}, \tag{31}$$

which has components with frequencies $jN\Omega$. Substitution of this periodic rotational torque and the periodic substructure forces in Eq. (30) into Eqs. (3a) and (6) yields the equations of motion for this rotor system with loading from N cyclically symmetric substructures [8].

The gyroscopic modal analysis described above is again applied to obtain the system response. As before, it allows one to separate the effects of absorbers on the two components of the response, namely rotor translational/tilting vibration and rotor rotational vibration.

The rotor rotation is affected only by the rotational modes. Because the rotational modes of three-dimensional CPVA systems are identical to those of planar CPVA systems [13], the optimal tuning of the absorbers for rotor rotational vibration reduction is the same as that derived in [8]. Specifically, the rotor rotation at frequency $jN\Omega$ can be eliminated by an absorber group with tuning order equal to $n_g = jN$. The design issues and effects of detuning described above also apply in this case.

For the translational-tilting vibrations, we consider the modal coordinate equation given in Eq. (11). With the frequency content of F_x and F_y given in Eq. (30), the modal coordinates $a_i(t)$ have steady-state responses with frequencies $(jN\pm1)\Omega$, and thus all degrees of freedom in the system response $\mathbf{z}(t)$ have those frequency components. For the translational-tilting modes, the modal deflections x_i and y_i have equal amplitudes, and y_i is 90° (for phase index $k=1$) or $-90°$ (for phase index $k=N-1$) phase-shifted relative to x_i [13]. Given these relationships and the previously discussed phase relationships of F_x and F_y, the response components of $x(t)$ and $y(t)$ associated with frequency $(jN-1)\Omega$ are $-90°$ phase-shifted relative to each other and have equal amplitudes, while the response components associated with frequency $(jN+1)\Omega$ are 90° phase-shifted with equal amplitudes. The amplitude and phase relations of the tilting response components $\nu(t)$ and $\eta(t)$ are the same. These phase relations indicate that the components of F_x and F_y corresponding to frequency $(jN+1)\Omega$ excite only the translational-tilting modes with phase index $k=1$, whereas the components corresponding to frequency $(jN-1)\Omega$ excite the translational-tilting modes with phase index $k=N-1$, which are the complex conjugates of the modes with phase index $k = 1$ [9,10,13].

Therefore, the steady-state responses of the rotor translational and tilting degrees of freedom are of the form

$$x(t) = \sum_{j=1}^{\infty}\{A_j\cos[(jN-1)\Omega t]+B_j\cos[(jN+1)\Omega t]\}, \tag{32a}$$

$$y(t) = \sum_{j=1}^{\infty}\{A_j\sin[(jN-1)\Omega t]-B_j\sin[(jN+1)\Omega t]\}, \tag{32b}$$

$$\nu(t)=\sum_{j=1}^{\infty}\{-C_j\ \sin[(jN-1)\Omega t]+D_j\sin[(jN+1)\Omega t]\},\tag{32c}$$

$$\eta(t)=\sum_{j=1}^{\infty}\{C_j\cos[(jN-1)\Omega t]+D_j\cos[(jN+1)\Omega t]\},\tag{32d}$$

with as yet unknown amplitudes A_j, B_j, C_j, and D_j. The phase relationships between the rotor translation and tilting in the translational-tilting modes derived in [13] require A_j and C_j to have the same sign, and B_j and D_j must also have the same sign. The absorber responses have the form

$$\hat{s}_g^{(i)}(t)=\sum_{j=1}^{\infty}\left\{E_{g,j}\sin\left[(jN-1)\Omega t-\beta_g^{(i)}\right]+H_{g,j}\sin\left[(jN+1)\Omega t+\beta_g^{(i)}\right]\right\},$$
$$i=1,2,\ldots,N_g,\quad g=1,2,\ldots,p.\tag{33}$$

Substitution of Eqs. (32) and (33) into Eq. (14) yields

$$\sum_{j=1}^{\infty}\left(\left\{(jN)^2(A_j+L_gC_j)+\left[(jN-1)^2-n_g^2\right]E_{g,j}\right\}\sin\left[(jN-1)\Omega t-\beta_g^{(i)}\right]\right.$$
$$\left.+\left\{(jN)^2(B_j+L_gD_j)-[(jN+1)^2-n_g^2]H_{g,j}\right\}\sin\left[(jN+1)\Omega t+\beta_g^{(i)}\right]\right)=0,$$
$$i=1,2,\ldots,N_g,\quad g=1,2,\ldots,p.\tag{34}$$

Projecting these equations onto their orthogonal time-harmonic components dictates that coefficients of the sine terms must vanish independently, giving

$$A_j+L_gC_j=\frac{n_g^2-(jN-1)^2}{(jN)^2}E_{g,j},$$
$$B_j+L_gD_j=-\frac{n_g^2-(jN+1)^2}{(jN)^2}H_{g,j},\tag{35a}$$

$$i=1,2,\ldots,N_g,\quad g=1,2,\ldots,p,\quad j=1,2,\ldots,\infty.\tag{35b}$$

The analysis for this case follows the single harmonic case analyzed earlier for excitation at frequency $m\Omega$, so further details are not given here. It is simply noted that for $L_g\neq 0$ (the absorber plane is not at the COM) the rotor translational and tilting amplitudes can be made zero, $A_j=C_j=0$, by tuning one absorber group with $n_g=jN-1$ and, similarly, $B_j=D_j=0$ when another absorber group has tuning order $n_g=jN+1$. The effects of detuning, and the design strategies described for the single harmonic case hold in this case as well.

6. Numerical example

A single absorber group with six absorbers and their parameter values given in Table 1 is considered to reduce the translational, tilting, and rotational vibrations of a rotor system under second rotor-order ($m=2$) lateral and torsional excitations. We consider the effects of the absorber tuning order n_g on the effectiveness in reducing rotor vibrations. The

Table 1
Parameters of second rotor-order lateral and torsional excitation reductions using a single absorber group.

Parameter	Value
Rotor mass, m_r (kg)	11
Rotor moment of inertia about shaft axis, J_r (kg m^2)	0.2
Rotor tilting moment of inertia, J_t (kg m^2)	2
Rotor translational stiffness, k_r (N/m)	1×10^9
Rotor tilting stiffness, K_t (N m)	1×10^9
Absorber mass, m_g (kg)	0.9
Distance between the center of mass and rotor, L_g (m)	0.5
Distance between center and absorber pivot, l_g (m)	0.0324–0.0441
Absorber radius, r (m)	0.01
Number of absorbers, N_g	6
Lateral force amplitude, F (N)	10
Torsional torque amplitude, T (N m)	1×10^{-3}
Lateral force offset distance from the COM, L (m)	0.5
Rotor speed, Ω (rpm)	1×10^3

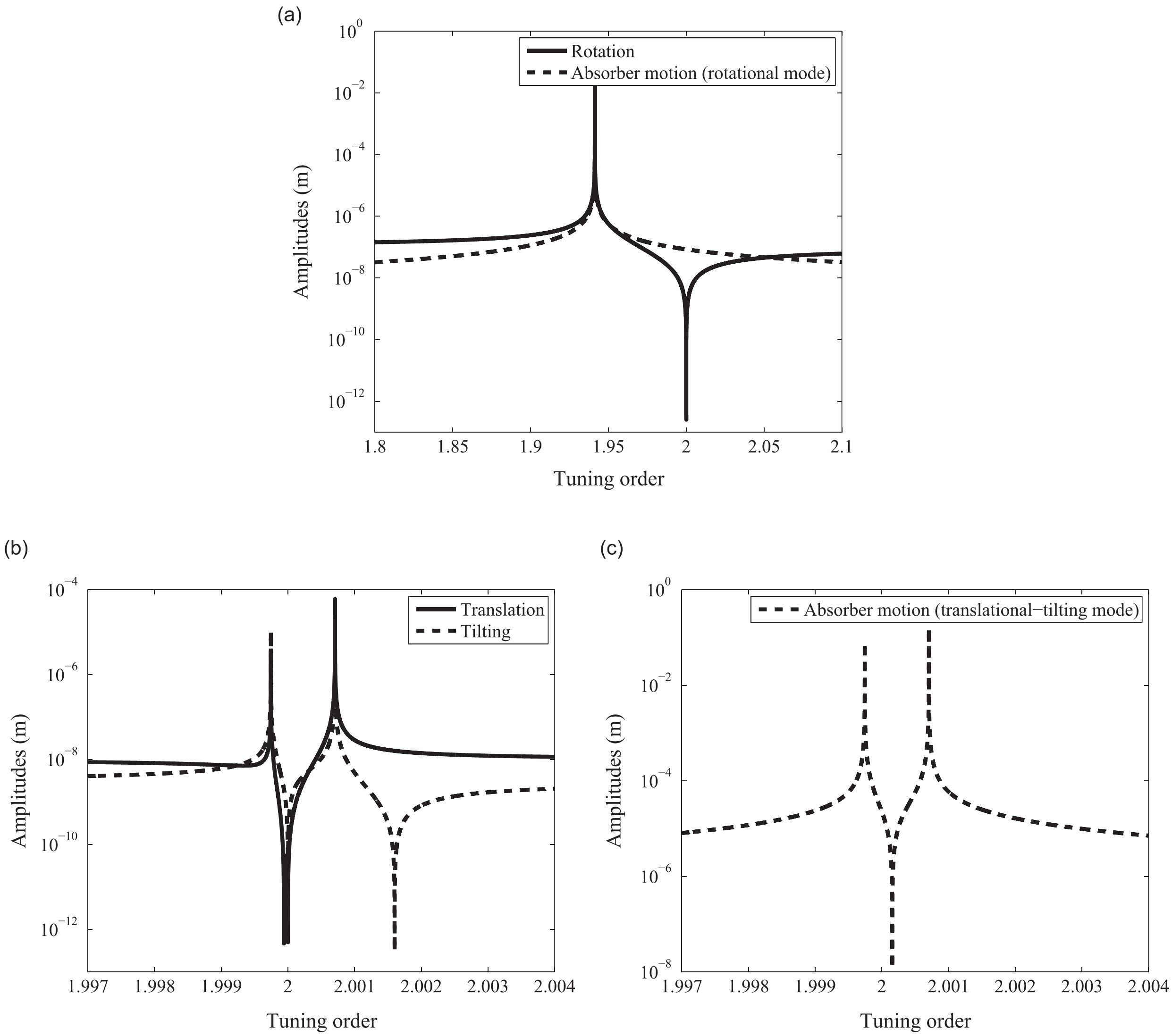

Fig. 4. Rotor and absorber amplitudes versus absorber tuning order for: (a) rotor rotation and (b,c) rotor translational and tilting. The solid curve in panel (a) is the rotor rotational amplitude and the dashed curve is the absorber amplitude for rotor rotation reduction. The solid and dashed curves in panel (b) are the rotor translational and tilting amplitudes, respectively. The dashed curve in panel (c) is the absorber amplitude for rotor translation and tilting reductions. The system parameters and external loads are given in Table 1.

distance between the rotor axis and the pivot position of each absorber l_g is tuned between 0.0324 m and 0.0441 m, so that the absorber tuning order n_g varies between 1.8 and 2.1.

The steady-state amplitudes of the rotor and absorber responses for rotor rotational vibration versus absorber tuning order n_g are shown in Fig. 4(a). Note that both the rotor and absorber responses exhibit a resonance, which occurs at the order corresponding to the frequency in Eq. (9). The rotor response has an anti-resonance at the exact tuning point, that is, $n_g = m = 2$, which is the ideal tuning condition. Note that the resonance and anti-resonance are separated by a factor of $1/\sqrt{1+\delta}$ where δ is the ratio of total absorber inertia to rotor inertia, and thus these are close in practice since δ is small, typically 0.1. A strategy for absorber tuning for this case is to make the absorbers slightly overtuned, that is, selecting $n_g > m$ by a small amount. This results in some residual rotational vibration of the rotor, yet it moves the system away from resonance and results in lower absorber amplitudes compared to ideal tuning. If the neglected damping were included, it will reduce the absorber amplitudes, which keeps the absorbers vibrating in the approximately linear range for a greater range of torque amplitude. This reduces the motivation to overtune to compensate for nonlinearity. Near the anti-resonance, the rotor rotation and absorber motion are 180^{o} out-of-phase, as predicted for the rotational modes derived in [13], resulting in the absorber canceling the rotational vibration.

The effects of damping on this response are well known: it limits the resonance peak and lifts the anti-resonance up from zero amplitude. In fact, the resonance peak is limited by a combination of rotor rotational and absorber damping, because this resonance involves an out-of-phase motion of the absorbers and rotor, while the anti-resonance is affected primarily by the absorber damping because the rotor is essentially vibration free at this point. Because these absorbers are order tuned, that is, because the ratio of m/n_g is fixed by the nature of the rotor loads and the absorber hardware, one typically designs the absorbers to be lightly damped, and therefore effective without ever encountering the resonance.

Fig. 4(b) and (c) shows the steady-state amplitudes of the rotor translational and tilting motions and absorber responses, respectively, versus the absorber tuning order n_g. Again, in this case an anti-resonance occurs at the exact tuning order $n_g = m = 2$, at which the absorber amplitude is finite. For these responses, a resonance occurs on both sides of the ideal tuning point, and extremely close to it, so that detuning the absorbers is not a feasible approach to avoiding resonance. However, just as in the case of rotor rotational motion, the rotor damping in translation and tilting will be the primary source to limit these resonance peaks, and the absorber damping will primarily affect the anti-resonance amplitude. Therefore, the feasibility of using these absorbers to limit rotor translation and rotation relies on some level of damping in those responses, since otherwise the tuning approach is highly sensitive to nearby resonances. In fact, we examined a wide range of rotor and absorber parameter values and found these nearby resonances to be unavoidable.

The absorbers will have separate harmonic components that correspond to addressing the rotational and translational/tilting vibrations, and that their combined response will be a linear combination of these separate effects.

7. Conclusions

We have considered the application of CPVAs to reducing vibrations of a rigid rotor that is free to translate, tilt, and rotate due to various types of rotor-order excitations. The model accounts for multiple groups of mutually identical CPVAs in order to address the vibrations arising from these loads. The use of gyroscopic modal analysis along with the special modal properties of this three-dimensional system lead to the useful separation of the vibration analysis for rotor translational-tilting and rotor rotational degrees of freedom, which makes the analysis tractable.

It is first shown how one can use a single group of CPVAs, placed symmetrically around the rotor in a plane offset from the COM and the plane of the loading, to simultaneously reduce rotor translational, tilting, and rotational vibrations caused by external periodic forces and rotor torque with frequency $m\Omega$. These absorbers must be tuned near to order m. Although the vibrations are theoretically eliminated in the case of exact tuning, practical aspects of the design often require detuning of the absorbers, and those design considerations are described.

The results are generalized to the case of periodic forces and torque that result from N substructures that are cyclically placed around the rotor. In this case the rotor rotational vibrations at order j are reduced (eliminated) by an absorber group with tuning close to (exactly at) order jN, whereas two absorber groups tuned close to (exactly at) orders $jN \pm 1$ are required to simultaneously reduce (eliminate) the rotor translational and tilting vibrations. Design considerations and the conditions required to maintain small absorber amplitudes, as required for the linear analysis to apply, are provided.

The results are based on a model that ignores damping and small imperfections, such as internal mistuning among the sets of absorbers and nonidentical loading arising from the substructures. As such, they provide useful guidelines for the initial selection of absorber parameters in terms of tuning and sizing, but nonlinear, damped, and mistuned models may be required when designing absorbers for specific applications. Consideration of the effects of damping, especially how it affects absorber performance and the potential difficulties of resonances near the ideal operating conditions, is an especially important next step in this topic of research.

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