TOPOLOGICAL 4-MANIFOLDS: THE DISC EMBEDDING THEOREM AND BEYOND
TECH TOPOLOGY SUMMER SCHOOL

ARUNIMA RAY

1. Personnel and resources

The TAs for this course are:

- Patrick Orson (ETHZ) https://people.math.ethz.ch/∼porson/
- Benjamin Ruppik (MPIM) https://ben300694.github.io/

Videos of the lectures and further resources will be made available at https://ttss.math.gatech.edu/ray-mini-course/

2. Outline of the course

The field of 4-manifold topology was revolutionised in the 1980s by concurrent work of Freedman and Donaldson, which established a vast disparity between the behaviour of smooth vs topological 4-manifolds. Soon after, Frank Quinn expanded on Freedman's techniques and established fundamental tools for topological 4-manifolds, such as transversality. The goal of this mini-course is to introduce the work of Freedman and Quinn, focussing primarily on topological 4-manifolds, explaining the motivation and context for their work, illustrating the key tools and ideas, and outlining potential future directions.

My hope is to convince you of the following claims:

- The topological category is not as scary as you might think.
- We know a lot about topological 4-manifolds, but there is much more to be done.
- Smooth manifold topologists should be interested in these topics.
- Smooth manifold topologists can also contribute to answers to the big open questions for topological 4-manifolds.

In particular, as explained in the Stipsicz lectures (https://ttss.math.gatech.edu/stipsicz-mini-course/), the results in the topological category, especially topological classification results, provide the context for several smooth results. For example, in order to find exotic smooth structures, we need on one hand to know when two 4-manifolds are homeomorphic (for which we need topological tools) and on the other hand to know when they fail to be diffeomorphic (for which we need smooth invariants).

2.1. **Recorded talks.** In the recorded talks made available before the summer school, I will explain the proof of the following theorem.

Theorem 1. *Let M and N be closed, smooth, simply connected 4-manifolds which have the same intersection form. Then they are homeomorphic.*

This has two main steps. The first is a result of Wall, saying that such 4-manifolds are smoothly h-*cobordant*. The second is Freedman's result, that smooth, simply connected h-cobordisms are topologically

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trivial. The proof of Freedman's result will use the disc embedding theorem, in order to find Whitney discs along which one may perform the Whitney trick. All of these notions will be explained.

2.2. **Live talks.** In the first two live talks, I will introduce the work of Quinn, establishing fundamental tools in topological 4-manifold topology. The key technical result is *handle straightening*. The result often needed in practice is that noncompact, connected topological 4-manifolds are smoothable.

The final talk will briefly describe related topics which we did not explore in the previous talks, but focus on open problems and potential future directions.