ECE 3077, Summer 2014

Homework #5

Due Thursday June 26, in class

Reading: B&T 3.1–3.5

1. Using you class notes, prepare a 1–2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. When you go to the grocery store you are equally likely to find 0 or 1 customers in front of you when you go to checkout. If there is a customer in front of you, the time it takes them to checkout is exponentially distributed with parameter λ. What is the cdf of your waiting time?

3. Metal meter sticks are manufactured on a production line where the true length can be modeled as a normal random variable X with mean 1 m (one meter) and standard deviation 2 mm (two millimeters).

   (a) What is the probability that the length of the meter stick X is within ± 0.5 mm of 1 m?

   (b) Suppose that the meter sticks are checked as they come off the production line and are rejected if they are more than 1 mm short or long. What is the pdf (in units of meters) of the length X meter sticks after rejecting those too long or two short? That is, with $A = \{\text{stick was within tolerance}\}$, find $f_{X|A}(x)$.

   (c) After the rejection process, what is the probability that the meter stick is within ± 0.5 mm of 1 m?

4. Romeo and Juliet have arranged to meet at a certain place at a certain time. They are both delayed by an indeterminate amount of time; we will use R to model Romeo's delay and J to model Juliet's delay (both in units of hours), and assume that these random variables have the following joint pdf:

$$f_{R,J}(r,j) = \begin{cases} \lambda_1 \lambda_2 \, \mathrm{e}^{-\lambda_1 r - \lambda_2 j} & r, j \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the probability that Romeo and Juliet arrive within 15 minutes of one another?

   (b) What is the expected absolute difference in their arrival time? That is, calculate $\mathrm{E}[|R-J|]$. (Hint: start by breaking the integral onto two pieces along the diagonal $x < y$ and $x \geq y$...)

5. Let X, Y be uniformly distributed on the unit square:

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Calculate $\mathrm{E}[X^2 + Y^2]$.

   (b) Calculate the expected distance of (X, Y) to the origin; that is, calculate $\mathrm{E}[\sqrt{X^2 + Y^2}]$. The integral(s) are hard to compute in this case, so feel free to use Wolfram Alpha or other external resources. But **carefully document** exactly how you arrived at your answer.

6. Suppose that X and Y are independent random variables both of which are uniformly distributed on the interval $[0, 1]$. Find the expectation of the distance between X and Y. That is, compute $\mathrm{E}[|X - Y|]$. (Hint: the answer is not 1/2 and definitely not 0.)

7. Describe how you would generate samples of random variables (X, Y) that are uniformly distributed inside the unit circle,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Write a MATLAB function `rand_uniform_circle` that takes a integer N, and returns N realizations $(X_1, Y_1), \ldots, (X_N, Y_N)$ distributed as above. Turn in your code and a scatter plot (using the `scatter` command) for $N = 1000$.

8. **Monte Carlo Integration**. Integrals can be hard to compute, especially for functions of many variables over complicated domains. In this problem, we get a first look at using random numbers to help us approximate integrals.

   (a) Let $g(x)$ be a function of a single random variable, and let X be a uniform random variables on the interval $[a, b]$:

$$X \sim \text{Uniform}([a, b]), \quad \text{or} \quad f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

   Now let Z be the random variables $Z = (b - a)g(X)$. What is $\mathrm{E}[Z]$?

   (b) Let $g(x_1, x_2, \ldots, x_n)$ be a function of n variables, and let $X_1, X_2, \ldots, X_n$ be independent uniform random variables on possibly different intervals:

$$X_i \sim \text{Uniform}([a_i, b_i]),$$

   for fixed $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ with $a_i < b_i$. Let

$$Z = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)\, g(X_1, X_2, \ldots, X_n).$$

   What is $\mathrm{E}[Z]$?

(c) Write a MATLAB script that generates a sequence of uniform random variables and uses them to estimate

$$\int_{x_1=6/5}^{9/5} \int_{x_2=1/2}^{9/10} \int_{x_3=0}^{1} \int_{x_4=1/5}^{13/10} \sqrt{17 - 4x_1^2 - 2x_2^2 + 3x_3^2 - x_4^2} \, \mathrm{d}x_4 \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1.$$

Note that a single realization of $X \sim \mathrm{Uniform}([a, b])$ can be generated in MATLAB using `x=(b-a)*rand(1)+a`.

As always, turn in your code and an estimate that you are confident is correct to 2 decimal points.