**ECE 3077, Summer 2014**

**Homework #7**

**Due Thursday July 17, in class**

1. Using you class notes, prepare a 1–2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Romeo and Juliet start dating and have plans to meet every night at a certain time and location. Romeo knows that Juliet tends to be tardy in a somewhat unpredictable way; on any given night, her arrival time $X$ can be modeled as uniformly distributed on $[0, D]$, where $D$ is her "intrinsic lateness parameter". Romeo would like to know $D$ so he knows just what kind of person he is dating. At the beginning of their relationship, not knowing too much, he takes $D$ to be uniformly distributed between 0 and 1 hour:
$$f_D(d) = \begin{cases} 1 & 0 \le d \le 1 \\ 0 & \text{else} \end{cases}.$$
Suppose that on their first date, Juliet arrives at time $X = x$. How should Romeo use this information to update the distribution for $D$?

3. In this problem, we will expand on the jelly bean example from the class notes. If you get stuck, I suggest looking at problem 30 on page 194 of B&T. Here, the proportion of red jelly beans is unknown, and we model it as a random variable $P$:
$$f_P(p) = \begin{cases} 1 & 0 \le p \le 1, \\ 0 & \text{otherwise.} \end{cases}$$
Let $X$ be a Bernoulli random variable that specifies whether a red or white jelly bean was drawn:
$$X = \begin{cases} 1 & \text{a red jelly bean is drawn,} \\ 0 & \text{a white jelly bean is drawn,} \end{cases}$$
and so for a particular $P = p$,
$$\mathrm{P}\left(X = 1 \mid P = p\right) = p, \qquad \mathrm{P}\left(X = 0 \mid P = p\right) = 1 - p.$$
We will consider drawing a sequence of jelly beans, captured by $X_1, X_2, \ldots, X_N$, which are *independent conditioned on* $P$. This means that if we draw out two jelly beans, we can write their conditional joint pmf as
$$p_{X_1,X_2|P}(x_1, x_2|p) = p_{X_1|P}(x_1|p)\, p_{X_2|P}(x_2|p),$$
and more generally if we draw out $N$ jelly beans,
$$p_{X_1,\ldots,X_N|P}(x_1, \ldots, x_n|p) = p_{X_1|P}(x_1|p) \cdots p_{X_N|P}(x_n|p).$$

(a) Say we draw out two jelly beans, the first is red (and so $X_1 = 1$) and the second is white ($X_2 = 0$). Find the conditional pdf $f_{P|X_1,X_2}(p \mid 1, 0)$ for $P$ given these observations, and plot it.

(b) Say we draw out $N$ jelly beans capture by the sequence $X_1, X_2, \ldots, X_N$. Find an expression for the conditional pdf $f_{P|X_1,\ldots,X_N}(p \mid x_1, \ldots, x_N)$. Your expression should involve $p$, $N$, the number of red jelly beans encountered $K = \sum_{i=1}^{N} X_i$ and the Beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x.$$

(c) Suppose we draw out 15 jelly beans and see the following sequence for the $X_i$:

$$1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1$$

(so $K = 10$). Use MATLAB to plot the conditional pdf for $P$ given these observations. Note that the `beta` command in MATLAB can be used to compute $B(\alpha, \beta)$. What is the conditional mean and variance of $P$ given this particular string of observations?

4. In this problem we will expand on the iPhone example from class. Say we test $N$ iPhones; their lifetimes are independent random variables $Y_1, Y_2, \ldots, Y_N$ conditioned on unknown parameter $\lambda$, modeled as

$$f_\Lambda(\lambda) = \begin{cases} 2 & 1/2 \le \lambda \le 1 \\ 0 & \text{otherwise,} \end{cases} \qquad f_{Y_i|\Lambda}(y_i \mid \lambda) = \lambda \mathrm{e}^{-\lambda y_i}, \quad y_i \ge 0.$$

Given a particular value of $\Lambda = \lambda$, then, the conditional joint density for $Y_1, Y_2, \ldots, Y_N$ is

$$f_{Y_1,\ldots,Y_N|\Lambda}(y_1, \ldots, y_N \mid \lambda) = f_{Y_1|\Lambda}(y_1|\lambda) f_{Y_2|\Lambda}(y_2|\lambda) \cdots f_{Y_N|\Lambda}(y_N|\lambda).$$

(a) Find an expression for the conditional pdf $f_{Y_1,\ldots,Y_N|\Lambda}(y_1, \ldots, y_N \mid \lambda)$.

(b) Find an expression for the unconditional pdf $f_{Y_1,Y_2,\ldots,Y_N}(y_1, \ldots, y_N)$. (See the definition of the incomplete gamma function below.)

(c) Suppose we observe the lifetime of each of the iPhones, and record the results as $Y_1 = y_1,\ Y_2 = y_2, \ldots, Y_n = y_n$. Find an expression for the conditional pdf of $\Lambda$ $f_{\Lambda|Y_1,\ldots,Y_N}(\lambda|y_1, \ldots, y_n)$ given these observations.

(d) The file `iPhoneTimes.mat` contains a vector $Y$ with 25 entries. We will treat these as 25 independent observations $y_i$, $i = 1, \ldots, 25$. Using MATLAB, plot $f_{\Lambda|Y_1,\ldots,Y_{25}}(\lambda \mid y_1, \ldots, y_{25})$ as a function of $\lambda$ for these particular $y_1, y_2, \ldots, y_{25}$.

(e) The entries of `Y` are indeed independent realizations of an exponential random variable with some $\lambda_0$ in the range $[1/2, 1]$. Use your results from the previous part to make an educated guess as to what $\lambda_0$ might have been.

There is one thing to know about which will make your life easier on this problem. The *incomplete gamma function* is defined as

$$\gamma(x,a) = \frac{1}{(a-1)!}\int_0^x t^{a-1}\mathrm{e}^{-t}\mathrm{d}t \tag{1}$$

where $a \geq 0$ is an integer. Although it does not have a closed-form expression, this is a useful quantity in many places in science and engineering, and hence it does have a dedicated MATLAB function `gammainc(x,a)`. It is a fact that

$$\int_0^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t = (a-1)!,$$

so $\gamma(\infty, a) = 1$ for all integers $a \geq 0$.

With a change of variable in the integral in (1), we have

$$\int_{x_1}^{x_2} t^n \mathrm{e}^{-bt}\mathrm{d}t = n! \cdot b^{-n-1}\left[\gamma(bx_2, n+1) - \gamma(bx_1, n+1)\right].$$

5. Let $X \sim \text{Uniform}([0,1])$.

   (a) Find a function $g(x)$ such that $Y = g(X)$ is distributed $Y \sim \text{Exp}(\lambda)$.

   (b) Find a function $g(x)$ such that $Y = g(X)$ has a *Rayleigh* distribution with pdf

   $$f_Y(y) = \frac{y}{\sigma^2}\mathrm{e}^{-y^2/2\sigma^2}, \quad y \geq 0$$

   and cdf

   $$F_Y(u) = \int_0^u \frac{y}{\sigma^2}\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y = 1 - \mathrm{e}^{-u^2/2\sigma^2}, \quad u \geq 0,$$

   (above, $F_y(u) = 0$ for $u < 0$).

6. Let $X$ and $Y$ be independent random variables that are uniformly distributed on the interval $[0,1]$. Set $Z = XY$.

   (a) Make 3 sketches, each with the unit square in the plane overlaid with the curves $xy = 1/4$, $xy = 1/2$, $xy = 1$.

   (b) Find the cdf for $Z$, $F_Z(z) = \mathrm{P}(Z \leq z)$. [Hint: For a given value of $z$, calculate the area corresponding to the region $xy < z$ using the sketches above.] Use MATLAB to plot it. (And turn in your plot, obviously.) Be careful around the point $z = 0$.

   (c) Find the pdf for $Z$. Use MATLAB to plot it. Be careful around the point $z = 0$. What happens as $z \to 0$ and $z \to 1$?

   (d) Write a MATLAB script that computes a discretized version of the pdf. It should take a number of bins $N$ and return a vector $f$ of length $N$ such that

   $$f(n) = \mathrm{P}\left(\frac{n-1}{N} \leq Z \leq \frac{n}{N}\right), \quad n = 1, \ldots, N,$$

   This should be just a few lines of code, given your answer to (b). Turn in a stem plot of $f$ for $N = 40$ with the horizontal axis labeled appropriately (the labels should be between 0 and 1). Again, be careful with the first bin.

(e) In MATLAB, generate 10,000 independent realization of $Z$. (The command `rand(10000,1)` will create a vector of $10,000$ independent uniform random variables on $[0,1]$.) Make a histogram with 40 bins using

```
bc = 0.0125:.025:.9875;
hist(Z, bc);
```

(The `bc` variable above specifies the bin centers.) Plot the result and compare to your plot in part (d).

7. Let $X \sim \text{Normal}(0,1)$. Find the pdf $f_Y(y)$ for

   (a) $Y = X^3$,

   (b) $Y = X^4$.