FINAL EXAM (ECE3075 Summer 2004): 07/29/04

There are six problems, worth a total of 120 points. You should first finish the first five problems. Then work towards the sixth problem for extra credit.

Problem 1 (20%)

Let X and Y be two independent standardized (zero mean, unit variance) Gaussian random variables. Define two additional random variables, $U=aX+bY$, and $V=aX-bY$, with a and b being two positive scaling constants.

(a) Find the means and variances of U and V;

(b) Find the correlation $E[UV]$ and correlation coefficient of U and V.

(c) Use the Jacobian transformation to find the joint probability density function of U and V, and express it in the standard bivariate Gaussian density form [Recall the Jocobian matrix J is defined as a 2x2 matrix with $\partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u,\ \partial y/\partial v$ as its elements];

(d) Make sure your answers in Parts (a) and (b) agree with the pdf in Part (c);

(e) Find the marginal probability density functions, $f_U(u)$ and $f_V(v)$. Are the marginal densities depending on the value of the correlation coefficient?

(f) Under what conditions can U and V be independent? State your reasons.

Problem 2 (20%)

A passenger can take two different bus lines to work. Both lines stop at the same bus stop. Let X and Y be independent random variables representing the times of arrival for the bus of line 1 and line 2, respectively. It is known that X and Y have exponential densities with a mean of 5 and 10 minutes, respectively. Let Z be the random variable for the time the passenger will wait for a bus, i.e. $Z=\min(X, Y)$. [Hint: $f_X(x) = K_x e^{-a_x x} u(x)$, and $f_Y(y) = K_y e^{-a_y y} u(y)$, with $u(.)$ a step function]

(a) Find the pairs of values of (K_x, a_x) and (K_y, a_y) to make $f_X(x)$ and $f_Y(y)$ valid densities, and with the corresponding means of 5 and 10 minutes;

(b) Find $P(Z>5)$, the probability that after 5 minutes no bus has arrived yet. Interpret your result and make sure the answer is reasonable [Hint: divide the *(x,y)* plane into two regions: $R_1 = \{Y \ge X\}$ and $R_2 = \{X > Y\}$];

(c) Find the probability density function of Z [Hint: it is obtained by taking the derivative of the corresponding distribution function, $F_Z(z) = P(Z \le z)$];

(d) What is the average time, $E[Z]$, the passenger will wait at the bus stop?

(e) Compare $E[Z]$ with $E[X]$ and $E[Y]$, and make sure your answer in Part (d) is intuitively reasonable. Check this answer with the result in Part (b).

Problem 3 (20%)

You play with a friend the following game: you flip a coin provided by your friend and if the result is Head you win \$1 while if it is a Tail you lost \$1 (i.e. you win -\$1). Let X_i be the random variable describing the amount you win at the *i*-th coin flip. You play the game *n*=2,500 times. Let *V* be the number of winning games, and $S=\sum_{i=1}^{n} X_i$ be the random variable representing the total winning amount after these plays. First assume a fair coin is used, i.e. *p=P(X=1)= P(X=-1)=0.5*.

(a) *V* is known to have a Binomial distribution with $P_n(V=v)={}_nC_v p^v(1-p)^{n-v}$, where ${}_nC_v=\frac{n!}{v!(n-v)!}$, show that *E[V]=np*, and Var[*V*]=*np(1-p)*;

(b) According to the Law of Large Number, *V* is approximately normal when *n* is large, find the approximate probability density functions of *V* and *S*;

(c) What is the probability that at the end of the game you lost more than \$300, i.e. find *P(S<-300)* [Hint: an approximation is fine, no need to be exact];

(d) At the end of the game you lost \$300. You suspect your friend cheated and the coin is biased. Estimate the value of the success rate, *p=P(X=1)*;

(e) Construct a 99% confidence interval for the unknown parameter, *p*;

(f) Do you think the coin is fair? Why?

Problem 4 (20%)

It is desirable to estimate the difference between the mean sale prices for all residential properties sold during 1983 in two neighborhoods D and E of a mid-sized Florida city. A random set of n_D sale prices for neighborhood D was collected with X_{Di} being the *i*-th sample with unknown mean μ_D and variance σ_D^2. Another random set of n_E sale prices for neighborhood E was also collected with X_{Ej} being the *j*-th sample with unknown mean μ_E and variance σ_E^2. All the random samples are assumed to be mutually independent of each other. We are interested in estimating the mean difference $\mu_Y=\mu_D-\mu_E$.

(a) Define a new random variable *Y*, which serve as a good estimate for μ_Y, if corresponding random samples are used to form a sample statistic for *Y*;

(b) Find the mean and standard deviation σ_Y of *Y*, is *Y* an unbiased estimate?

(c) Give a sample statistic which serve as a unbiased point estimates for σ_Y^2;

(d) From the actual samples in surveys we have obtained the following set of measurements: $n_D=30$ (sample size), $n_E=40$ (sample size), $\bar{x}_D=\$52,356$ (sample mean), (sample mean), $\tilde{s}_{D2}=\$10,572$ (sample standard deviation), $\bar{x}_E=\$66,491$ (sample mean), and $\tilde{s}_{E2}=\$14,264$ (sample standard deviation). Construct a 95% confidence interval for μ_Y [Hint: Find the approximate large sample density for the corresponding sample statistic];

(e) If we want to test a null hypothesis, H_0: $\mu_Y=0$ against an alternative hypothesis, H_1: $\mu_Y < 0$, construct a statistical test. Based on the data in part(c), decide if we should reject H_0 at a 95% confidence level, why?

Problem 5 (20%)

A zero-mean random process, *X(t),* is wide-sense stationary, with autocorrelation function $R_X(\tau) = 10e^{-2|\tau|} - 5e^{-4|\tau|}$. Another zero-mean, stationary noise process, *N(t)*, has a band-pass spectral density: $S_N(\omega) = 1/5, \text{ for } 10\pi \leq |\omega| \leq 20\pi, \text{ and } S_N(\omega) = 0 \text{ elsewhere}$. Assume *X(t)* and *N(t)* are independent. Define a noisy process: *Y(t)=aX(t-q)+N(t)*, where *a* is a constant much smaller than 1, *q* represents a round-trip time delay in return signal acquisition to measure the distance to a target using the signal, *X(t)*.

(a) Find the spectral density, $S_X(\omega)$, and $S_X(s)$ (with $s \triangleq j\omega$) of process *X(t)*;

(b) Is $S_X(\omega)$ a rational spectrum? If so, find all the poles and zeroes of $S_X(s)$?

(c) Find $\int_{-\infty}^{\infty} S_X(\omega) d\omega$, and verify $\int_{-\infty}^{\infty} S_X(\omega) d\omega = R_X(0) = E[X^2]$ [Hint: the integration can be accomplished by the Residue Theorem];

(d) Find the mean-square value of the noise process, *N(t)*;

(e) Find the autocorrelation, $R_Y(\tau)$, and mean-square value, $R_Y(0)$, of *Y(t)* ;

(f) Compute the cross-correlation function, $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$, and the cross-spectral density, $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$;

(g) Describe a way to estimate the delay, *q*.

Problem 6 (Extra Credit: +20%)

Consider a low pass RC circuit that characterize a linear and time-invariance (LTI) system, with a frequency response: *H(s)=b/(s+b)* with b=1/RC representing a positive decaying constant. It can be shown that the impulse response of the LTI system is simply $h(t) = be^{-bt},\ t \geq 0$, and $h(t) = 0,\ t < 0$. A random signal, *X(t)*, passes through the system will produce a random output, *Y(t)*, computed as the following convolution integral: $Y(t) = \int_0^{\infty} X(t-s)h(s) ds$. In general *X(t)* is not zero mean.

(a) If *X(t)* is a white noise with $S_Y(f) = S_0, -\infty < f < \infty$, show that $R_X(\tau) = S_0 \delta(\tau)$;

(b) Find $E[Y(t)]$ and $E[Y(t)Y(t+\tau)]$, and show that *Y(t)* is wide-sense stationary;

(c) If *X(t)* is non-white with $R_X(\tau) = Ke^{-\beta|\tau|}, \beta > 0$, find $S_Y(\omega)$. Determine the value of *K* such that $S_Y(0) = S_0$, i.e. same dc value as the above white noise case;

(d) Given the autocorrelation of *X(t)* in Part (c), find the autocorrelation of the corresponding system output, $R_Y(\tau) = E[Y(t)Y(t+\tau)]$. Verify that when $\beta \to \infty$, $R_Y(\tau)$ will converge to the solution in Part (b), an intuitive consequence;

(e) If $\beta \gg b$, i.e. the input spectrum has a much wider bandwidth than that (constant *b*) of the low pass LTI system response, show that the above white noise approximation in Part (d) still holds, another important result.

[Paper copy of PDF Table for Normal Density is attached for your reference]