ECE3075 - Random Signals

Chapter 1: Introduction

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Course Information

- **Subject:** Random Signal and Systems
- **Prerequisite:** ECE2025 and 3770 (or MATH 3770 or CEE 3770)
- **Background Expected**
  - Fundamentals on Calculus, Physics, and Digital Signal Processing
- **Grades: Homework (25%), Two Quizzes (15+20%), Final (40%)**
- **Tools Expected**
  - Basic analytical skills, and MATLAB programming (optional but helpful)
  - Textbooks and reading assignments: your main source of learning
  - Class Lectures: exploring beyond the textbooks
  - Homework: hand-on and get-your-hands-dirty exercises
- **Website:** http://users.ece.gatech.edu/~chl/ECE3075.sum04/
  - Check frequently for class notes, updates, and homework assignments

Teaching Philosophy

- **Textbook+Lecture: Main Source of Learning**
  - Detailed white-board explanation of textbook materials
  - Intuitive interpretation of mathematical formulations
  - Supplemental examples and illustrations
- **Lecture Notes: Summary of weekly lecture materials**
- **Homework: Hands-on Exercises**
  - About 10 sets to complement lectures
  - Assign on Thursdays, collect the next Thursday, no late turn-in
  - Solutions given on the following Tuesdays
- **Office Hours: Individual Q&A**
  - 90 minutes after each class, more by appointment when needed
- **Course Website: Update news and e-correspondences**

Grading Policy

- **Homework (25%)**
  - About 10 sets to complement lectures
  - Assign on Thursdays, collect the next Thursday
  - Solutions given on the following Tuesdays
  - No grade given to late homework
- **Exams (75%): calculator allowed, no notes, no books**
  - Quiz #1 (15%): tentatively scheduled on June 8 for 90 minutes
  - Quiz #2 (20%): tentatively scheduled on July 1 for 90 minutes
  - Final (40%): July 29 for 3 hours

Course Materials

- **Textbook**
  - **G. R. Cooper and C. D. McGillem, *Probabilistic Methods of Signal and System Analysis*, 3rd Edition, Oxford Press, 1999**
- **Topics to be covered: intensive learning**
  - Introduction to Probability
  - Random Variables
  - Several Random Variables
  - Elements of Statistics
  - Random Processes
  - Correlation Functions
  - Spectral Density
  - Linear Systems with Random Inputs

Some Real-World Examples

- Odds and Statistics
  1. What is the winning chance for lotto 6? Or quick 4?
  2. What is the probability of having 20 high card points in a bridge hand? Or a small slam hand between partners?
  3. Random Experiments: Will this new drug work?
- Characterization of signals and environments
  1. How to design wireless communication systems?
  2. How to design multimedia storage and display systems?
- Basic Tools
  - Counting, set theory and measuring of random “events”
  - Formal probability function of random variables

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Why Studying Probability?

- Random Input Signals
- Random Noise
- Random System Reliability
- Quality Control (ISyE)
- Information Theory
- Simulation and Sampling
- Others
  - Statistical Thermodynamics and Physics

Elementary Probability

- Definition: What is probability?
  - Exact definition not important
- The important concept of "Events" (Table 1-1)
  1. Random Experiments and "outcomes" of "trials"
     - discrete or continuous in values
  2. Description of outcomes and random events
     - measurement of simple and composite events
- Two Closely Related Approaches
  - Relative-Frequency Approach: Physical Analysis
  - Axiomatic Approach: Mathematical Analysis

The Relative-Frequency Approach

- Intuitive and related to real-world examples
  1. Count random "events"
  2. Relate events to sets
  3. Measure and compare sets
- In contrast to the Axiomatic Approach
  – Similarity; measuring sets and their relationships
  – Difference: physical vs. mathematical
  – More later, need both to characterize a problem
- Closely related to Descriptive Statistics
  – To be discussed in Chapter 4

Counting Simple Events

- Winning chance for lotto 6
  1. How many numbers? 42 or 48?
  2. Factorial operation: $n!=n*(n-1)*\ldots*2*1$
  3. How to draw the number balls? Put back after each draw?
  4. Does the ordering of the draws matter?
  5. Are the number balls biased?
- Exactly 3 heads appearing after flipping a coin 5 times
  1. How is the coin tossed? What is the surface?
  2. Does the five tosses identical? The concept of permutation
  3. Is the coin biased? In what way?
  4. Does the person and way of tossing affecting the outcome?
  5. Choosing operation: $C(n, m)=n!/[m!(n-m)!]$

Counting Complex Events

- Throwing two dices (joint event)
  1. Probability of having equal to or more than seven points?
  2. Are the two dices identical?
  3. Is each number on a dice showing up with equal chance?
  4. How to design the dices to maximize the winning chances for a dealer? How to beat the odds as a player?
- Drawing five cards in a Poker game
  1. Is a flush more likely than two pairs?
  2. Any change with with only cards of 8 points or higher?
  3. Is the deck of cards biased? In what way?
  4. How to design a deck of cards in favor of the dealer or players? Is the game fair? Can it be improved?

Classroom Illustrations (I): Counting

- Blackboard Illustrations
  1. Section 1-4
  2. Table 1-2
  3. Exercise 1-4.1
  4. Exercise 1-4.2
- Other Illustrations
  1. Coin Flipping: head/tail, repeated trials
  2. Throwing two dices, repeated trials
  3. Drawing cards from a deck, repeated trials
  4. Lotto drawing

Elementary Set Theory

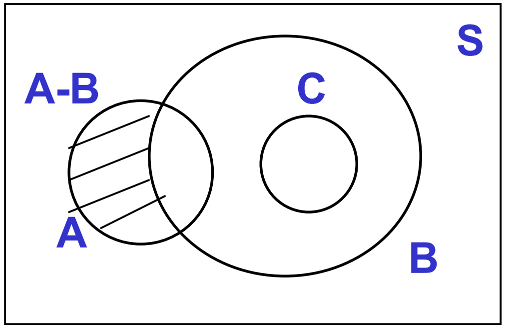
- **Set: a collection of objects known as elements**

$$A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \text{ and } \alpha_i \in A$$

- **Universe (Space): the set of all elements (alphabets)**

$$I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}, \quad \| I \| = \infty$$

- **Cardinality: Size of the universe, can be uncountable**
- **Graphical representation:** ***Venn Diagram***
- **Null or Empty Set:** ϕ
- **Subset and Membership:** $C \subset B$
  - If $||S||=n$ then S has $2^{**}n$ subsets
  - *Exercise*: prove by counting/permutation



Set Operations and Corresponding Laws

- **Union (Sum):** $A \cup B$
- **Intersection (Product):** $A \cap B$
- **Equality:** $A = B$ if and only if $A \subset B$ and $B \subset A$
- **Complement :** $\overline{A}$ such that $A \cup \overline{A} = U$ and $A \cap \overline{A} = \phi$
- **Difference :** $A - B = A \cap \overline{B} = A - (A \cap B)$
- **Combination of set operations:** $C = A \cap \bigcup_{i=1}^{N} B_i$
- **Mutually exclusive or disjoint sets:** $A \cap B = \phi$
- **Laws of set operations**
  - Commutative Law: $A \cap B = B \cap A$
  - Associative Law: $(A \cup B) \cup C = A \cup (B \cup C)$
  - Distributive Law: $A \cap (B \cup C) = (A \cap B) \cap (A \cup C)$

Classroom Illustrations (II): Sets

- Blackboard Illustrations
  1. Section 1-5
  2. Exercise 1-5.1
  3. Exercise 1-5.2

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The Axiomatic Approach

- Sample Space (Probability Space): S
  - collection of all possible observed outcomes
- Sample Event: including the null event ϕ
- σ-field Ω : set of all possible event sets
- Probability Function (Measurable) $P:\Omega\rightarrow[0,1]$
- Three Fundamental Axioms:
  1. $P(A)\geq 0 \qquad \forall A\in\Omega\ (A\subset S)$
  2. $P(S)=1 \qquad S\in\Omega$ (certain event)
  3. If $A\cap B=\phi$ then $P(A\cup B)=P(A)+P(B), \forall A,B\in\Omega$

Some Real-World Examples

- Sample Space:
  - $\Omega_c = \{x\text{: } x \text{ is the height of a person}\}$
  - $\Omega_d = \{(y, z)\text{: } y \text{ is the age and } z \text{ is his/her resident city}\}$
- Sample Event:
  - $A=\{x\text{: } x>200\text{cm}\}$
  - $B=\{x\text{: } 120\text{cm}<x<130\text{cm, and } x<70\text{cm}\}$
  - $C=\{(y, z)\text{: (teens, Atlanta)}\}$
  - $D=\{(y, z)\text{: (over 70; Japan)}\}$
- Set of all possible events Ω: collection of all sets like C
- Probability Function (Measurable): $P:\Omega\rightarrow[0,1]$
  - measuring A, B, C and D: computing $P(A)$, $P(B)$, $P(C)$, $P(D)$

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Classroom Illustration (III): Probability

- Blackboard Illustrations
  1. Section 1-6
  2. Exercise 1-6.1
  3. Exercise 1-6.2
- Other Illustrations: $S=\{x: x = \text{height of a person}\}$
  - $A=\{x: x>200\text{cm}\}$, $P(A)=(\text{\# taller than 2m})/(\text{total \#})$
  - $B=\{x: 120\text{cm}<x<130\text{cm and } x<70\text{cm}\}$ $B=E\cup F$
  - $C=\{x: x>120\text{cm and } x<130\text{cm}\}$ $C=E\cup F$

$$P(E\cup F)=P(E)+P(F)-P(E\cap F)$$

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Joint Events

- Outcomes of two or more simultaneous events
  - $A=\{x\text{: height } x>200\text{cm}\}$, $B=\{y\text{: } y \text{ is an NBA player}\}$
  1. Involving more sample spaces (product space)
  2. Involving the same sample space (more trials)
- Joint Sample Space: e.g. $S = S_1 \times S_2$
  - Joint σ-field: $\Omega = \Omega_1 \times \Omega_2$
  - Joint probability: $P : \Omega \rightarrow [0,1]$
  - Let $C=(A, B)$, how to evaluate $P(C)$?
- Marginal Probability: same as one single space

$P_1 : \Omega_1 \rightarrow [0,1]$ and $P_2 : \Omega_2 \rightarrow [0,1]$, but how to get P_1 and P_2 from P

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Conditional Events

- Modeling events in the presence of others
  - $A=\{x$: height $x>200$cm$\}$, $B=\{y$: y is an NBA player$\}$
  1. C=$\{z$: $z>200$cm given that z is an NBA player$\}$
  2. In other words: $P(C)=P(A|B)$
- Satisfying the Three Fundamental Axioms:
  1. $P(A|B)\geq 0 \quad \forall A\in\Omega\ (A\subset S)$
  2. $P(S|B)=1 \quad S\in\Omega$ (certain event)
  3. If $A_1\cap A_2=\phi$ then $P(A_1\cup A_2|B)=P(A_1|B)+P(A_2|B)$
- Conditional Probability: $P(A|B)=P(A,B)/P(B)$
- A and B can be from the same space: $P(A,B)=P(A\cap B)$

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Prior and Posterior Probabilities

- Prior vs. Posterior Events: in the same space
  - Prior Event *A* before any samples observed
  - Posterior Event *A* after observing some evidence *B*
  - *P(A), P(B|A), P(A|B): what can we say about them?*
- An Example Venn Diagram: Figure 1-7
  1. $S = A_1 \cup A_2 \cup \cdots \cup A_n$, A_i and A_j $(i \neq j)$ are disjoint
  2. $B = B \cap (A_1 \cup \cdots \cup A_n) = (B \cap A_1) \cup \cdots \cup (B \cap A_n)$
  3. $P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$
- Total Prob.: $P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)$

Bayes' Theorem

- Many Forms of Bayes's Theorem (items 2 and 3)
  1. $P(A \cap B) = P(B \mid A)P(A) = P(A \mid B)P(B)$
  2. We have: $P(A_i \mid B) = P(B \mid A_i)P(A_i) / P(B)$
  3. Or

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{[P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)]}$$

  4. Note the interchange role in item 1
- Any physical interpretation?
  – See Illustration on Exercise 1-7.1 and 1-7.2

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Statistical Independence

- Definition: two events *A* and *B* are independent if

$$P(A \cap B) = P(A) * P(B)$$

- Any physical interpretation?
  1. From Bayes' Theorem: $P(A \mid B) = P(A)$
  2. Observing *B* does not improve our inference on *A*
  3. No physical mechanisms to link *A* and *B* together
- Overall vs. Pair-wise Independence:
  – Important concept: equation (1-24) on page 27
  1. $P(A \cup B) = P(A) + P(B) - P(A) * P(B)$
  2. $P([A \cap (B \cup C)]) = P(A) * P(B \cup C)$

Classroom Illustrations (IV): Conditional Probability and Independence

- Blackboard Illustrations
  1. Section 1-7
  2. Tables 1-3
  3. Exercise 1-7.1
  4. Exercise 1-7.2
  5. Exercise 1-8.1
  6. Exercise 1-8.2

Classroom Illustrations (V): Combined or Joint Events

- Blackboard Illustrations
  1. Section 1-9
  2. Exercise 1-9.1
  3. Exercise 1-9.2

Bernoulli Trials and Applications

- Binary Events:

$$P(A) = P(\text{"success"}) = p, \; P(\bar{A}) = P(\text{"failure"}) = q = 1 - p$$

- How about k successes in n independent trials?
  - How many such possibilities: *binomial coefficient*

$${}_nC_k = \binom{n}{k} = \frac{1}{k!}[n*(n-1)\cdots*(n-k+1)] = \frac{n!}{k!(n-k)!}$$

$$p_n(k) = P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{(n-k)}$$

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Some Bernoulli Trials

- Binary Events:

$$P(\text{at most } K \text{ successes in } n \text{ trials}) = \sum_{i=0}^{K} \binom{n}{i} p^i q^{(n-i)}$$

- *DeMoivre-Laplace Theorem*

If $npq \gg 1$ and $|k - np|$ is in the order of $\sqrt{npq}$ then

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} \exp[-\frac{(k-np)^2}{2npq}]$$

Later we will see this is due to Gaussian Approximation.

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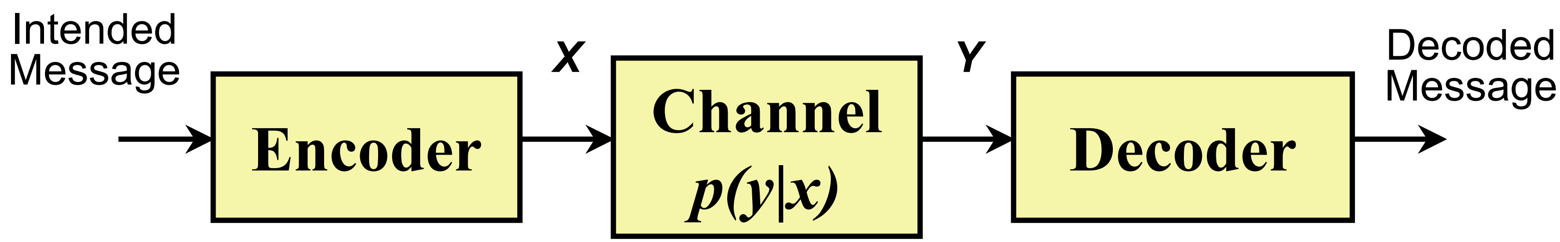
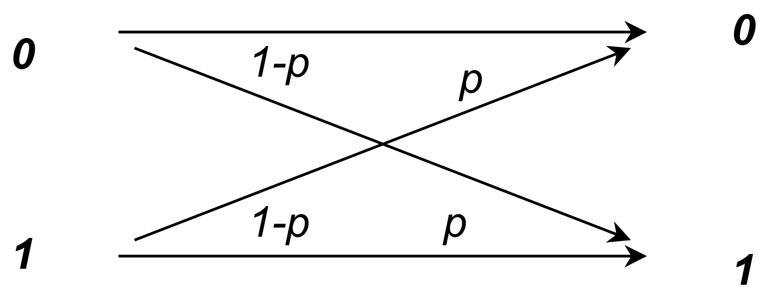
Noisy Channel Model

- **Shannon's Noisy Channel Model**



- **A Binary Symmetric Channel (Modem Application)**



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Classroom Illustrations (VI): Bernoulli Trials and Applications

- Communication and Signal Detection
  - Digital communication
  - Radar: binary integration, or detection
    - type I error: P(false detection)
    - type II error: P(false alarms)
  - System reliability and redundancy
- More Blackboard Illustrations
  1. Sections 1-10 and 1-11
  2. Exercise 1-10.1
  3. Exercise 1-10.2

Summary

- **Today's Class**
  - Information about ECE3075, Summer 2004
  - Basic Probability Theory
  - Course Web: http://www.ece.gatech.edu/~chl/ECE3075.sum04/
- **Reading Assignment and Exercises**
  - Cooper & McGillem, Chapter 1
- **Class Next Week**
  - Random variables