ECE3075 - Random Signals

Chapter 3: Several Random Variables

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Joint Distribution Functions

- Probability Distribution Function of two r.v.'s X and Y

$$F(x, y) = P(X \le x, Y \le y)$$

- Some Important Properties

1. $0 \le F(x, y) \le 1, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $F(-\infty, y) = F(x, -\infty) = F(-\infty, -\infty) = 0,$ and $F(\infty, \infty) = 1$
3. $F(x, y)$ is a non-decreasing function of either x or y, or both
4. $F(\infty, y) = F_Y(y)$ and $F(x, \infty) = F_X(x)$

- Illustration Examples
  - Tossing two coins and observing the outcomes
  - Figures 3-1 and 3-2

Joint Density Functions

- Joint pdf of *X* and *Y*, if exists, completely specifies all conditional and marginal probabilities (later)

$$f(x, y) = \partial^2 F(x, y) / \partial x \partial y$$

- Some Important Properties

1. $f(x, y) \geq 0, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$
3. $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) dv du$
4. $P(x_1 \leq x < x_2, y_1 \leq y < y_2) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$
5. $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$

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Expectation of Random Functions

- Expectation: $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$
- Correlation between two random variables

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$$

- An Illustration Example: Uniform distribution

$$f(x,y) = 1/[(x_2 - x_1)(y_2 - y_1)],\ x_1 \le x \le x_2, y_1 \le y \le y_2$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy/[(x_2 - x_1)(y_2 - y_1)]dxdy = (x_2 + x_1)(y_2 + y_1)/4$$

$$f_X(x) = \int_{y_1}^{y_2} f(x,y)dy = 1/(x_2 - x_1),\ x_1 \le x \le x_2;$$

$$f_Y(y) = \int_{x_1}^{x_2} f(x,y)dx = 1/(y_2 - y_1),\ y_1 \le y \le y_2;$$

- More Examples: Exercises 3-1.1 and 3-1.2

Conditional and Marginal Probabilities

- *Conditional* PDF and pdf (continuous variable case):

$$F_X(x \mid Y \leq y) = P(X \leq x \mid M) = P(X \leq x, M) / P(M) = F(x, y) / F_Y(y)$$

$$F_X(x \mid y_1 \leq Y \leq y_2) = [F(x, y_2) - F(x, y_1)] / [F_Y(y_2) - F_Y(y_1)]$$

$$f_X(x \mid Y = y) = \lim_{\Delta y \to 0} [F(x, y + \Delta y) - F(x, y)] / [F_Y(y + \Delta y) - F_Y(y)]$$
$$= f(x, y) / f_Y(y) \text{ and similarly } f_Y(y \mid X = x) = f(x, y) / f_X(x)$$

- Bayes' Theorem (with valid non-zero denominators)

$$f(y \mid x) = [f(x \mid y) * f(y)] / f(x) \quad f(x \mid y) = [f(y \mid x) * f(x)] / f(y)$$

Marginal Probability

- *Marginal* PDF and pdf (continuous variable case):

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \text{ and similarly } f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

- Textbook Example:

$$f(x,y) = \tfrac{6}{5}(1 - x^2 y),\ 0 \le x \le 1,\ 0 \le y \le 1$$

$$f_X(x) = \tfrac{6}{5}(1 - x^2/2),\ 0 \le x \le 1 \text{ and } f_Y(y) = \tfrac{6}{5}(1 - y/3),\ 0 \le y \le 1$$

$$f_X(x \mid y) = (1 - x^2 y)/(1 - y/3),\ 0 \le x \le 1\ [0 \le y \le 1]$$

$$f_Y(y \mid x) = (1 - x^2 y)/(1 - x^2/2),\ 0 \le y \le 1\ [0 \le x \le 1]$$

Example: Signal with Additive Noise

- Given observed noisy signal $Y=X+N$ with N as additive noise, want to estimate the signal X

$$f(x|y) = \frac{f(y|x)f(x)}{f_Y(y)} = \frac{f_N(y-x)f_X(x)}{\int_{-\infty}^{\infty} f_N(y-x)f_X(x)dx}$$

- Marginal exponential signal and Gaussian noise

$$f_X(x) = b\exp(-bx), \quad x \ge 0 \qquad f_N(n) = \frac{1}{\sqrt{2\pi\sigma_n^2}}\exp(-\frac{1}{2\sigma_n^2}n^2)$$

$$f_Y(y) = \int_0^{\infty} \frac{b}{\sqrt{2\pi\sigma_n^2}}\exp[-\frac{1}{2\sigma_n^2}(y-x)^2]\exp(-bx)dx = b\exp(-by + \frac{b^2\sigma_n^2}{2})Q(-\frac{y-b\sigma_n^2}{\sigma_n})$$

$$f(x|y) = \frac{b}{f_Y(y)\sqrt{2\pi\sigma_n^2}}\exp\{-\frac{1}{2\sigma_n^2}[x^2 - 2(y-b\sigma_n^2)x + y^2]\}, \quad x \ge 0$$

Example: Additive Noise (Cont.)

- Given observed y what is an "optimal" estimate of x

$\hat{x} = \max_x f(x \mid y)$ or solving for $\partial f(x \mid y)/\partial x = 0$

we get $2\hat{x} - 2(y - b\sigma_n^2) = 0$ or $\hat{x} = y - b\sigma_n^2$, if $y > b\sigma_n^2$.

If $y < b\sigma_n^2$, then $\hat{x} = 0$. Another interesting fact is that if the noise level is relatively small, i.e. $\sigma_n^2 \to 0$, then $\hat{x} \to y$.

- Other definitions of "optimality" can also be used
- Textbook Illustrations
  - Exercises 3-2.1 and 3-2.2

Statistical Independence

- Necessary and Sufficient Condition: $f(x,y) = f_X(x) f_Y(y)$
- Conditional Expectation of correlation between X and Y

$$E(XY) = \int x f_X(x)dx * \int y f_Y(y)dy = E(X) * E(Y)$$

- Bayes's Theorem
  – prior information no longer helps for mutual inference

$$f(x|y) = f(x,y)/f_Y(y) = f_X(x)$$

$$f(y|x) = f(x,y)/f_X(x) = f_Y(y)$$

- Textbook Illustrations
  – Exercises 3-3.1 and 3-3.2 (Laplace density)

Properties of Correlation Coefficient

- Define standardized r.v.'s (zero mean, unity variance):

$$\xi = (X - \bar{X}) / \sigma_X \text{ and } \eta = (Y - \bar{Y}) / \sigma_Y \quad (\bar{\xi} = 0, \bar{\eta} = 0, \sigma_\xi^2 = 1, \sigma_\eta^2 = 1)$$

- Correlation coefficient (normalized covariance)

$$\rho = E[\xi\eta] \text{ and } E[(\xi \pm \eta)^2] = 2(1 \pm \rho) \geq 0 \Rightarrow -1 \leq \rho \leq 1$$

- Statistical independence implies $\rho = E[\xi\eta] = \bar{\xi}\bar{\eta} = 0$
- Converse: zero correlation does not imply statistical independence unless for Gaussian random variables
- Textbook Illustrations (for your exercise)
  - An example on pp. 133-134

More Properties of Covariance

- Random variables as functions of standardized r.v.'s

$$X = \sigma_X \xi + \bar{X} \text{ and } Y = \eta \sigma_Y + \bar{Y}$$

- Correlation between X and Y

$$E[XY] = E[\sigma_X \sigma_Y \xi \eta + \bar{X}\sigma_Y \eta + \bar{Y}\sigma_X \xi + \bar{X}\bar{Y}] = \rho \sigma_X \sigma_Y + \bar{X}\bar{Y}$$

- Random expectations

$$E[(X \pm Y)^2] = \sigma_X^2 + \sigma_Y^2 \pm 2\rho \sigma_X \sigma_Y + (\bar{X} \pm \bar{Y})^2$$

- Variance of $X+Y$ and $X-Y$

$$\sigma_{(X \pm Y)}^2 = Var[(X \pm Y)] = \sigma_X^2 + \sigma_Y^2 \pm 2\rho \sigma_X \sigma_Y$$

- Textbook Illustrations
  - Exercises 3-4.1 and 3-4.2

Bivariate Gaussian Density

- X and Y are jointly Gaussian with the density

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2 \sigma_X \sigma_Y \sqrt{1-\rho^2}}$$

$$\exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y}]\}$$

- Show marginal densities are Gaussian $f_X(x)$ $f_Y(y)$
- Show conditional densities are Gaussian

$$f_X(x|y) = f(x,y)/f(y) \text{ and } f_Y(y|x) = f(x,y)/f(x)$$

- If X and Y are uncorrelated $(\rho = 0 \Rightarrow$ Independence)

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2 \sigma_X \sigma_Y} \exp\{-\frac{1}{2}[(\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2]\} = f_X(x) f_Y(y)$$

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Density Function of *Z=X+Y* and *Z=X-Y*

- Sum of two random variables *Z=X+Y*

$$F_Z(z) = P(Z \le z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x,y)\,dx\,dy$$

- Special Case: If *X* and *Y* are independent

$$F_Z(z) = \int_{-\infty}^{\infty} f_Y(y)\left[\int_{-\infty}^{z-y} f_X(x)\,dx\right]dy$$

- Density Function (convolution integral)

$$f_Z(z) = dF_Z(z)/dz = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\,dy = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

- Difference of two random variables *W=X-Y*

$$f_W(w) = dF_W(w)/dw = \int_{-\infty}^{\infty} f_Y(y) f_X(w+y)\,dy = \int_{-\infty}^{\infty} f_X(x) f_Y(x-w)\,dx$$

- Textbook Illustration: pp. 137-138

Sum of Gaussian Random Variables

- Sum of two independent Gaussian r.v's $Z=X+Y$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^2 \sigma_X \sigma_Y} \exp\{-\frac{1}{2}[(\frac{x-\mu_X}{\sigma_X})^2 + (\frac{z-x-\mu_Y}{\sigma_Y})^2]\} dx$$

$$= \frac{1}{(\sqrt{2\pi(\sigma_X^2+\sigma_Y^2})} \exp\{-\frac{[z-(\mu_X+\mu_Y)]^2}{2(\sigma_X^2+\sigma_Y^2)}\}$$

- Z is also Gaussian with mean and variance equal to the sum of the means and variances of X and Y
- Reproducible densities: true even for correlated cases
- Law of large numbers: sum tends to Gaussian (Fig. 3-7)
- Textbook Illustration: Exercises 3-5.1 and 3-5.2

Density of a Function of Two R.V.'s

- X and Y have a joint pdf $f(x,y)$, want to find the joint density $g(z, w)$ of Z and W, with defined functions $Z = \varphi_1(X,Y)$ and $W = \varphi_2(X,Y)$ with an exisiting inverse relationship of the form $X = \Psi_1(Z,W)$ and $Y = \Psi_2(Z,W)$
- It is required that:

$$P(z_1 < z < z_2, w_1 < w < w_2) = P(x_1 < x < x_2, y_1 < y < y_2)$$

- Or: $$\int_{z_1}^{z_2}\int_{w_1}^{w_2} g(z,w)dwdz = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

$$= \int_{z_1}^{z_2}\int_{w_1}^{w_2} f(\Psi_1(z,w), \Psi_2(z,w))\,|J|\,dwdz$$

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Jacobian of Transformation

- Jacobian of transformation from *(X,Y)* onto *(Z,W)*

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} \ni g(z,w) = |J|\, f(\Psi_1(z,w), \Psi_2(z,w))$$

- Example (pp.143-147) and Exercises 3-6.1 and 3-6.2

$z = xy$ and $w = x$ (auxiliary), we $x = w$ and $y = z/w$

- We have: $$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1/w & -z/w^2 \end{vmatrix} = -\frac{1}{w}$$
- So

$$g(z,w) = \tfrac{1}{|w|} f(w, z/w) \text{ and } g_Z(z) = \int_{-\infty}^{\infty} \tfrac{1}{|w|} f(w, z/w)\,dw$$

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The Characteristic Function

- Convolution in time = product in frequency domain
- The Characteristic function (recall Fourier transform)

$$\phi(u) = E[e^{juX}] = \int_{-\infty}^{\infty} f(x) e^{jux} dx$$

- Inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u) e^{-jux} du$$

- Back to $Z=X+Y$ (convolution integral)

$$\phi_Z(u) = \int_{-\infty}^{\infty} e^{jzu} \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy dz \quad (\text{set } z = x + y)$$

$$= \int_{-\infty}^{\infty} e^{j(x+y)u} \int_{-\infty}^{\infty} f_Y(y) f_X(x) dy dx = \phi_X(u) \phi_Y(u)$$

An Example (pp. 149-150)

- X is uniform and Y is exponential, X, Y independent
- The Characteristic functions of X and Y, $Z=X+Y$

$$\phi_X(u)=\int_0^1 1*e^{jux}dx=\frac{e^{ju}-1}{ju} \text{ and } \phi_Y(u)=\int_0^\infty e^{-y}e^{juy}dy=\frac{1}{1-ju}$$

- Density of Z (standard Fourier transform techniques)

$$f_Z(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_X(u)\phi_Y(u)e^{-juz}du=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ju}-1}{ju(1-ju)}e^{-juz}du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ju(1-z)}}{ju(1-ju)}du-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{juz}}{ju(1-ju)}du$$

$$=1-e^{-z} \quad \text{when} \quad 0<z<1$$

$$=(e-1)e^{-z} \quad \text{when} \quad 1<z<\infty$$

An Application: Computing Moments

- Moments can be computed easily by differentiation

$$d\phi(u)/du\,|_{u=0} = \int_{-\infty}^{\infty} f(x)(jx)e^{jux}dx\,|_{u=0} = j\overline{X}$$

- Continuing differentiation:

$$E[X^n] = j^{-n}\left[d^n\phi(u)/du^n\right]|_{u=0}$$

- Extending to more random variables (e.g. two)

$$\phi(u,v) = E[e^{j(uX+vY)}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j(ux+vy)}dxdy$$

$$E[X^nY^m] = j^{-n}j^{-m}\left[\partial^{n+m}\phi(u,v)/\partial u^n\partial v^m\right]|_{u=v=0}$$

- Textbook illustrations: Exercises 3-7.1 and 3-7.2

Summary

- **Today's Class**
  - Several Random variables
- **Reading Assignments**
  - Cooper & McGillem, Chapter 3
- **Class Next Week**
  - Quiz #1 on 6/8/20 (Chapters 1-3)
  - Elements of Statistics on 6/10/04 (Chapter 4)