ECE3075 - Random Signals

Chapter 6: Correlation Functions

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Autocorrelation and Crosscorrelation

- Time Correlation: an important probability description

Ensemble 1: $X_1 = X(t_1)$ and $X_2 = X(t_2)$

Ensemble 2: $Y_3 = Y(t_3)$ and $Y_4 = Y(t_4)$

we have $R_{XX}(t_1, t_2) = E[X_1 X_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2) dx_2$

and $R_{XY}(t_1, t_3) = E[X_1 Y_3] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_3 f(x_1, y_3) dy_3$

- From general to more manageable definitions
  - assuming stationarity or ergodicty
  - many EE applications (Chapters 6-9)
- Illustration: Exercises 5-1.1 and 5-1.2

Correlation of Stationary Processes

- Assume *X(t)* is wide-sense stationary

$$R_X(t_1,t_2)=R_X(t_1+t,t_2+t)=R_X(0,t_2-t_1)=E[X(0)X(t_2-t_1)],$$

we have a common measure of similarity between $X(t)$ and $X(t+\tau)$:

$$R_X(\tau)=E[X(t)X(t+\tau)] \text{ (ensemble correlation, even function in } \tau)$$

- Correlation over time: time correlation function

$$\Re_x(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)x(t+\tau)dt = <[x(t)x(t+\tau)]>,$$

For an ergodic process: $\Re_x(\tau)=R_X(\tau)$

- Correlation coefficient: $\rho(\tau)=R_X(\tau)/R_X(0)=R_X(\tau)/\sigma_X^2$

An Example: Binary Process

- $X(t)$ varies between $-A$ and A (Figure 6-1)
  - note the difference between convolution integral and autocorrelation integral (Figure 6-2 is not correct)

$$R_X(\tau) = \begin{cases} A^2[1 - |\tau - 2kt_0| / t_a] & 0 \le |\tau| \le t_a, k \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

  - when the time difference s is small $X(t)$ and $X(t+s)$ tend to have the same value, and the autocorrelation is positive
  - when s gets large, the correlation becomes zero
  - the above function is periodic with a period of $2t_a$
- Illustrations: Exercises 6-2.1 and 6-2.2

Properties of Autocorrelation

- Important properties of wide-sense stationary process
  1. Energy/Power (mean squared value): $R_X(0) = E[X^2]$
  2. Even Function: $R_X(-\tau) = R_X(\tau)$
  3. Autocorrelation peaked at $\tau = 0$: $R_X(\tau) \le R_X(0)$
  4. If *X(t)* has a dc component, then $R_X(\tau)$ has also a dc part
  5. If *X(t)* is periodic, then $R_X(\tau)$ has also a periodic component (Figure 6-2 and Exercise 6-2.2)
  6. If *X(t)* is zero mean and ergodic, and not periodic, then

     $\lim_{\tau\to\infty} R_X(\tau) = 0$ (i.e. asymptotically independent)
  7. Constraints (spectral density property in Chapter 7): the Fourier transform is $\int R_X(\tau)\exp[-j\omega\tau]d\tau \ge 0 \quad \forall\omega$
  8. Joint pdf and $R_X(\tau)$
- Illustrations: Exercises 6-3.1 and 6-3.2

Measurement of Autocorrelation

- Estimating with time average based on sample function

$$\hat{R}_X(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)x(t+\tau)dt \text{ (a r. v.)}$$

$$\hat{R}_X(n\Delta t) = \frac{1}{N-n+1}\sum_{k=0}^{N-n} x(k\Delta t)x([k+n]\Delta t) \quad (n = 0,\ldots,M \ll N)$$

- Mean and Variance of autocorrelation estimate

$$E[\hat{R}_X(n\Delta t)] = \frac{1}{N-n+1}\sum_{k=0}^{N-n} E[x(k\Delta t)x([k+n]\Delta t)] = R_X(n\Delta t)$$

$$\mathrm{Var}[\hat{R}_X(n\Delta t)] \le \frac{2}{N}\sum_{k=-M}^{M} R_X^2(k\Delta t) \text{ (for your reference only)}$$

- Example on pp.222-225, and Exercises 6-4.1 and 6-4.2

Examples of Autocorrelation

1. Random binary process: uniform switching time
   - triangular autocorrelation (Figures 6-2, zero mean)
   - triangular autocorrelation (Figures 6-5, non-zero mean)
2. Binary process with random switching time
   - switching times are equally probable: exponential (Fig. 6-6)
   - random telegraph waveform (Figure 6-6)
   - non-differential process and discontinuity (infinite variance)

$R_X(\tau) = A^2 \exp[-\alpha \,|\, \tau \,|]$ (α is average number of intervals per second)

3. More in Chapters 7 and 8: with wideband noise input
   - output of bandpass filter (Figure 6-7(a))
   - output of ideal lowpass filter (Figure 6-7(b))
   - Exercises 6-5.1 and 6-5.2

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Crosscorrelation Functions

- Ensemble crosscorrelation function: jointly stationary

$$R_{XY}(\tau) = E[X(t_1)Y(t_1+\tau)] = E[X_1Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_2 f(x_1, y_2) dy_2$$

$$R_{YX}(\tau) = E[Y(t_1)X(t_1+\tau)] = E[Y_1X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} y_1 x_2 f(y_1, x_2) dx_2$$

- Time crosscorrelation function

$$\Re_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) y(t+\tau) dt$$

$$\Re_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t) x(t+\tau) dt$$

- Ergodic Process: $\Re_{xy}(\tau) = R_{XY}(\tau)$ and $\Re_{yx}(\tau) = R_{YX}(\tau)$

- Illustrations: Exercises 6-6.1 and 6-6.2

Properties of Crosscorrelation

- Important properties of wide-sense stationary process
  1. No strong physical interpretation: $R_{XY}(0) = R_{YX}(0)$
  2. Shift of X(t) in one direction=shift of Y(t) in opposite direction: $R_{YX}(-\tau) = R_{XY}(\tau)$
  3. Crosscorrelation is bounded: $|R_{XY}(\tau)| \le \sqrt{R_X(0) * R_Y(0)}$
  4. If *X(t)* and *Y(t)* are independent, then $R_{XY}(\tau) = R_{YX}(\tau)$
  5. If either *X(t)* or *Y(t)* is zero mean, then
     $$R_{XY}(\tau) = R_{YX}(\tau) = 0, \ \forall \tau$$
  6. If $\dot{X}(t)$ is the derivative process (in time) of *X(t)*
     $$R_{X\dot{X}}(\tau) = dR_{XX}(\tau)/d\tau \text{ and } R_{\dot{X}\dot{X}}(\tau) = dR_{X\dot{X}}(\tau)/d\tau = d^2 R_{XX}(\tau)/d\tau^2$$
     $$\text{with } \dot{X}(t) = \lim_{e \to 0}[X(t+e) - X(t)]/e$$
  7. More discussion in Chapters 7 and 8
- Illustrations: Exercises 6-7.1 and 6-7.2

Examples of Crosscorrelation

- Sum and difference of random processes: $Z(t) = X(t) \pm Y(t)$

$$Z_1 = X_1 + Y_1 = X(t) \pm Y(t), Z_2 = X_2 + Y_2 = X(t+\tau) \pm Y(t+\tau)$$

$$R_Z(\tau) = E[Z_1 Z_2] = E[X_1 X_2 + Y_1 Y_2 \pm X_1 Y_2 \pm Y_1 X_2]$$

$$= R_X(\tau) + R_Y(\tau) \pm R_{XY}(\tau) \pm R_{YX}(\tau)$$

- If *X(t)* and *Y(t)* are independent: $R_Z(\tau) = R_X(\tau) + R_Y(\tau)$
- Stereo Transformation: $U(t) = X(t) + Y(t), V(t) = X(t) - Y(t)$

$$R_{UV}(\tau) = E[U(t)V(t+\tau)] = R_X(\tau) + R_{YX}(\tau) - R_{XY}(\tau) - R_Y(\tau)$$

$$R_{VU}(\tau) = E[V(t)U(t+\tau)] = R_X(\tau) - R_{YX}(\tau) + R_{XY}(\tau) - R_Y(\tau)$$

If $X(t)$ and $Y(t)$ are zero mean and independent

$$\Rightarrow R_{UV}(\tau) = R_{VU}(\tau) = R_X(\tau) - R_Y(\tau)$$

An Application of Crosscorrelation

- Extraction of small signals from noisy observations:
  - radar and sonar signals: $Y(t) = aX(t - \tau_1) + N(t)$

$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = aR_X(\tau - \tau_1) + R_{XN}(\tau) = aR_X(\tau - \tau_1)$

[$a \leq 1$ is the oveall gain, is the round trip delay,

$N(t)$ is zero mean and independent of $X(t)$]

- By computing crosscorrelation, we obtain an estimate of the autocorrelation of the signal *X(t),* which is known to peak at a zero shift, which implies the overall trip delay can be estimated by peak-picking the crosscorrelation function, then obtaining the distance to the target
- MATLAB examples, and Exercises 6-8.1 and 6-8.2

Correlation Matrices for Sample Functions

- Define the random vector: $\mathbf{X} = [X(t_1), \ldots, X(t_N)]^T$

$$\mathbf{R}_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^T] = E\begin{bmatrix} X(t_1)X(t_1) & X(t_1)X(t_2) & \cdots & X(t_1)X(t_N) \\ X(t_2)X(t_1) & X(t_2)X(t_2) & \cdots & X(t_2)X(t_N) \\ \vdots & \vdots & \vdots & \vdots \\ X(t_N)X(t_1) & X(t_N)X(t_2) & \cdots & X(t_N)X(t_N) \end{bmatrix}$$

or assuming wide sense stationarity, even-spaced sampling in time

$$\mathbf{R}_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^T] = E\begin{bmatrix} R_X(0) & R_X(1) & \cdots & R_X(N-1) \\ R_X(1) & R_X(0) & \cdots & R_X(N-2) \\ \vdots & \vdots & \vdots & \vdots \\ R_X(N-1) & R_X(N-2) & \cdots & R_X(0) \end{bmatrix}$$

Covariance Matrices

- Define the random vector: $\mathbf{X} = [X(t_1), \ldots, X(t_N)]^T$

$$\mathbf{\Lambda_X} = E[(\mathbf{X}-\bar{\mathbf{X}})(\mathbf{X}-\bar{\mathbf{X}})^T] = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{N1}\sigma_2\sigma_1 & \rho_{N1}\sigma_N\sigma_1 & \cdots & \sigma_N^2 \end{bmatrix}$$

or for wide-sense stationary processes, we have a Toplitz matrix

$$\mathbf{\Lambda_X} = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & \cdots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{bmatrix} = \sigma^2 \Sigma_{\mathbf{X}} \text{ (a correlation matrix)}$$

Crosscorrelation Matrices for Sample Functions

- Define: $\mathbf{X} = [X_1(t), \ldots, X_N(t)]^T$ and $\mathbf{Y} = [Y_1(t), \ldots, Y_N(t)]^T$

Assuming wide sense stationarity, even-spaced sampling in time

$$\mathbf{R}_{\mathbf{XY}}(\tau) = E[\mathbf{X}(t)\mathbf{Y}^T(t+\tau)] = \begin{bmatrix} R_{11}(\tau) & R_{12}(\tau) & \cdots & R_{1N}(\tau) \\ R_{21}(\tau) & R_{22}(\tau) & \cdots & R_{2N}(\tau) \\ \vdots & \vdots & \vdots & \vdots \\ R_{NN}(\tau) & R_{N2}(\tau) & \cdots & R_{NN}(\tau) \end{bmatrix}$$

with $R_{ij}(\tau) = E[X_i(t)Y_j(t+\tau)]$, $1 \le i \le N$ and $1 \le j \le N$

- Non-symmetric matrices can also be defined
- Used to characterize input-output relations for linear systems with random inputs (in Chapters 8 and 9)
- Multivariate Gaussian densities (Equation 6-54)

Summary

- **Today's Class**
  - Correlation Functions
- **Quiz #2 on 7/1/04**
  - Chapters 4-6
- **Reading Assignments**
  - Cooper & McGillem, Chapter 6
- **Class Next Week**
  - Chapter 7