ECE3075 - Random Signals

Chapter 7: Spectral Density

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Frequency-Domain Analysis

- From time-domain to frequency-domain methods
  - non-random signals, y(t), we have the Fourier transform

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \quad \text{or } F_Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt$$

(both amplitude and phase density functions in ω or f)

- Random Signals: Autocorrelation and Spectral Density
  - direct Fourier transform often does not exit, the condition of absolutely summable (integrable) does not always exit even for wide-sense stationary processes
  - from one-sided to two-sided Laplace transforms

$$F(s) = \int_{0}^{\infty} y(t) e^{-st} dt \quad \text{(with a build-in convergence factor } s\text{)}$$

Regularity Conditions

- Requirements for defining FT for random processes

1. Finite Time: $X_T(t) = X(t), |t| \leq T < \infty$, and =0 otherwise

2. Square Integrable: $\int_{-\infty}^{\infty} |X_T(t)|^2 dt \quad (T < \infty)$

we have $F_X(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt$ (want $E[|F_X(\omega)|^2 < \infty$)

3. Parseval Theorem: $\int_{-T}^{T} |X_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_X(\omega)|^2 d\omega$

[recall the general form: $\int_{-\infty}^{\infty} f(t) g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G(-\omega) d\omega$]

- Energy or power implications: some requirements
  - average power as a function of frequency
  - expectation of average power (per unit bandwidth)

Spectral Density & Fourier Transform

- Ensemble average equals to time average

$$E[\frac{1}{2T}\int_{-T}^{T} X_T^2(t)dt] = E[\frac{1}{4\pi T}\int_{-\infty}^{\infty} F_X(\omega)F_X(-\omega)d\omega] =,$$

$$< \overline{X}^2 >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \overline{X}^2 dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[|F_X(\omega)|^2]}{2T} d\omega$$

- For stationary processes: Power Density Spectrum

$$E[X^2] =< \overline{X}^2 >= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \int_{-\infty}^{\infty} S_X(f)df$$

$$S_X(\omega) = \lim_{T\to\infty} E[|F_X(\omega)|^2]/2T \quad \text{(Spectral Density)}$$

- Illustrations: p. 261 and Exercises 7-2.1 and 7-2.2

Properties of Spectral Density

- Important properties of spectral densities
  - Real, positive and even function of the frequency
- Rational polynomial form (conjugate poles and zeroes):

$$S_X(\omega) = \frac{S_0(\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_2\omega^2 + a_0)}{\omega^{2m} + b_{2m-2}\omega^{2m-2} + \cdots + b_2\omega^2 + b_0} \quad (m > n)$$

- Non-rational form:

$$S_X(\omega) = (\sin 5\omega / 5\omega)^2 \quad \text{(for random binary signal)}$$

- Continuous vs. non-continuous spectral densities
  - Signals with dc and periodic components in time

An Important Example (Figure 7-1)

- Sinusoidal signal: $X(t) = A + B\sin(2\pi f_0 t + \theta)$

  A, B and f_0 are constants and θ is uniform over 0 and 2π

$$F_X(f) = \int_{-\infty}^{\infty} \{\mathrm{rect}(t/2T) * [A + B\sin(2\pi f_0 t)]\} e^{-j2\pi ft} dt$$

$$= 2T\mathrm{sinc}(2Tf) \otimes [A\delta(f) + \tfrac{B}{2}\delta(f + f_0)e^{-j\theta} + \tfrac{B}{2}\delta(f - f_0)e^{j\theta}]$$

$$= 2AT\mathrm{sinc}(2Tf) + BT\{\mathrm{sinc}[2\mathrm{T}(f + f_0)]e^{-j\theta} + \mathrm{sinc}[2\mathrm{T}(f - f_0)]e^{j\theta}]$$

- Time average (using Eq. (7-18) since θ is U[0, 2π])

$$\int_0^{2\pi} (e^{jn\theta} / 2\pi) d\theta = 0, \ \forall n$$

$$E[|F_X(f)|^2] = 4A^2T^2\mathrm{sinc}^2(Tf) +$$

$$B^2T^2\{\mathrm{sinc}^2[2\mathrm{T}(f + f_0)]e^{-j\theta} + \mathrm{sinc}^2[2\mathrm{T}(f - f_0)]e^{j\theta}]$$

An Important Example (Continued)

- Spectral Density
  - energy concentration at center frequency (FM)

$$S_X(f) = \lim_{T\to\infty} E[|F_X(f)|^2] \text{ (and using } \int_{-\infty}^{\infty} [\sin^2(at)]/t^2 dt = |a|\pi)$$

we have $S_X(f) = A^2\delta(f) + (B^2/4)[\delta(f+f_0) + \delta(f-f_0)]$

- Mean square average

$$E[X^2] = \int_{-\infty}^{\infty} S_X(f)df = A^2 + B^2/2 \text{ (= ensemble average)}$$

- Another example (Figures 7-2 and 7-3)
  - spread pulse train (in communication)
- Other examples: Exercises 7-3.1 and 7-3.2

A Numerical Example

- Signal: periodic rectangular pulse train: A is U[0, $2\mu_Y$]

$$Y(t) = A,\ \ -t_1 \le t - kt_0 \le t_1,\ \text{and}\ Y(t) = 0,\ \text{otherwise}\ (t_1(.01) < t_0(.1),\ k\text{: integer})$$

$$\text{we have } Y(f) = \int_{-\infty}^{\infty} [\sum_k \text{rect}(\text{t} - kt_0/2\text{t}_1)] e^{-j2\pi ft} dt = \frac{A}{t_1} \text{sinc}(ft_1) \sum_k \delta(f - kt_0 / t_1)$$

- Spectral density: with amplitude mean and variance :

$$S_X(f) = [\frac{1}{2t_1} \text{sinc}(\frac{f}{2t_1})]^2 [\frac{\sigma_Y^2}{t_0} + \frac{\mu_Y^2}{t_0^2} \sum_k \delta(f - \frac{ft_0}{2t_1} k)]$$

- Spectral density of the differential process

$$S_{\dot{X}}(f) = \lim_{T \to \infty} \frac{1}{2T} E[| j2\pi f F_X(f) * (-j2\pi f) F_X(-f)] = (2\pi f)^2 S_X(f)$$

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Spectral Density: Complex Frequency

- Rational Spectra: pole-zero configuration

$$S_X(\omega) = \frac{10(\omega^2+5)}{\omega^4+10\omega^2+24} \text{ or } S_X(s) = \frac{-10(s^2-5)}{s^4-10s^2+24} \quad (\omega^2=-s^2)$$

$$\text{Poles}: s = \pm 2, \pm\sqrt{6}, \text{ and zeroes}: s = \pm\sqrt{5} \text{ (all on the } j\omega \text{ axis)}$$

- Complex poles and zeroes

$$S_X(\omega) = \frac{\omega^2(\omega^2+25)}{\omega^6-33\omega^4+463\omega^2+7569} \text{ or } S_X(s) = \frac{-s^2(s^2-25)}{s^6+33s^4+463s^2-7569}$$

$$\text{Poles}: s = \pm 3, \pm(2 \pm j5), \text{ and zeroes}: s = 0, 0, \pm 5 \text{ (on the complex plane)}$$

- Example on pp.273, and Exercises 7-4.1 and 7-4.2

Mean-Square Values from Spectral Density

- Mean-square is proportional to the area under the spectral density (note: the pdf definition)

$$E[X^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{c(s)c(-s)}{d(s)d(-s)} ds \text{ (Table 7-1)}$$

$c(s)$ and $d(s)$ has only roots in the left half plane,

and $d(s)$ has no root on the $j\omega$-axis (imaginary axis)

- General case (integration in complex plane): Fig. 7-7

$$E[X^2] = \sum(\text{residues at left half plane poles}), \text{ residue } \triangleq [(s+\alpha)S_X(s)]|_{s=-\alpha}$$

- MATLAB example on pp. 279-280
- Illustration: Exercises 7-5.1 and 7-5.2

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Spectral Density & Autocorrelation

- Wiener-Khinchine Relation

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau, \quad \text{and } R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} d\omega$$

- An example: exponentially decayed function

$$R_X(\tau) = A\exp[-\beta|\tau|],\ S_X(\omega) = \frac{2A\beta}{(\omega^2+\beta^2)} = \frac{A}{j(\omega - j\beta)} - \frac{A}{j(\omega + j\beta)}$$

- Autocorrelation is an even function

$$S_X(\omega) = 2\int_{0}^{\infty} R_X(\tau)\cos(\omega\tau) d\tau, \text{ and } R_X(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} S_X(\omega)\cos(\omega\tau) d\omega$$

- Verify inverse: from spectral density to autocorrelation
- Textbook Illustration: Exercises 7-6.1 and 7-6.2

White Noise

- Noise that is white: equal energy over all frequencies
  - hypothetical case implies infinite power

$$S_X(\omega) = S_0 \Rightarrow R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0 e^{-j\omega\tau}\, d\omega = S_0 \delta(\tau)$$

- Bandlimited white noise (fictitious Figure 7-9)

$$S_X(f) = S_0 \mathrm{rect}(f/2W) \Rightarrow R_X(\tau) = \int_{-W}^{W} S_0 e^{j2\pi f\tau}\, df = 2WS_0 \mathrm{sinc}(2W\tau)$$

  - mean-square value is finite (a nice approximation)

$$E[X^2] = 2WS_0, \text{ or } R_X(0) = 2WS_0$$

- Recall Sampling Theorem: Nyquist sampling rate
- Textbook Illustration: Exercises 7-7.1 and 7-7.2

Cross-Spectral Density

- Cross-correlation function and cross-spectral density

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[F_X(-\omega)F_Y(\omega)]}{2T}, S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[F_Y(-\omega)F_X(\omega)]}{2T}$$

- Wiener-Khinchine relation extension

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau, R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega$$

- Two-sided Laplace transform: an example

$$S_{XY}(s) = \frac{-96}{s^2+2s-8} = \frac{16}{s+4} - \frac{16}{s-2}$$

$$R_{XY}(\tau) = 16e^{-4\tau}(\tau > 0), R_{XY}(\tau) = 16e^{2\tau}(\tau < 0)$$

$$S_{YX}(\omega) = S_{XY}^*(\omega)$$

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

- Illustrations: Exercises 7-8.1 and 7-8.2

Spectral Density Estimation

- Using Autocorrelation Estimate (Section 7-9)
  - from a single sample by time averaging (Eq. (6.14))
  - windowing operation (rectangular or Hamming window)
  - Fourier transform of the windowed auto-correlation estimation is computed by convolution over frequency
  - MATLAB exercises (corspec.m on pp. 298-301)
- Using Periodogram Estimate (Section 7-10)
  - multiple short segments of the observed sample function
  - averaging over the corresponding spectral densities, and resulting in a smoothed spectrum (Eq. (7-76))
  - MATLAB exercises (perspec.m on pp. 302-307)
- Detail with not be covered (more in advanced courses)

Examples of Spectral Density

- Example 1: rectangular pulse (Figure 7-17(a))

$$f(t) = A * \text{rect}(t/t_1),\ F(\omega) = At_1 \text{sinc}(t_1\omega/2\pi)$$

$$S_X(\omega) = A^2 * t_1 \text{sinc}^2(t_1\omega/2\pi)$$

- Example 2: raised cosine (or Hamming, Figure 7-17(b))

$$f(t) = \frac{B}{2}(1+\cos\frac{2\pi t}{t_1}) * \text{rect}(\frac{t}{2t_1}), F(\omega) = \frac{Bt_1}{2}\text{sinc}(\frac{\omega t_1}{2})[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}]$$

$$S_X(\omega) = \frac{B^2 t_1}{4}\text{sinc}^2(\frac{\omega t_1}{2})[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}]^2$$

- Spectral density plots: Figure 7-18
  - raised cosine is better in attenuation & bandwidth (lowpass)
  - bandpass spectral density (Figure 7-19)

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Summary

- **Today's Class**
  - Spectral Density
- **Reading Assignments**
  - Cooper & McGillem, Chapter 7
- **Class Next Week**
  - Chapter 8