Solution to Homework 1 (ECE3075 Summer 2004)

1. (10%) Work out Problem 1-1.2 in Cooper and Mcgillem.

   a) A: 20, B: 13, C: 13, D: 6, E: 46, F: 0, G: 2, H: 16, I: 22, J: 0, K: 0, L: 17, M: 8, N: 18, O: 23, P: 11, Q: 0, R:21, S: 22, T: 41, U: 9, V: 0, W: 3, X: 1, Y: 2, Z: 0, Space: 68 (depending on what's counted as a space)
   b) Most probable: Space; Next probable: E; Least probable six: F, J, K, Q, V, Z.

2. (10%) Work out Problem 1-4.2 in Cooper and Mcgillem.

   A pair of unbiased and fair dices are rolled, what is the probability of:
   a) Sum of the two is 11: P([5,6])+P([6,5])=1/36+1/36=1/18;
   b) Sum less than 5: P([1,1])+P([1,2])+P([2,1])+P([2,2])+P([1,3])+P([3,1])=1/6;
   c) Sum is even: P(S=2)+P(S=4)+P(S=6)+P(S=8)+P(S=10)+P(S=12) = 1/36 + 3/36 + 5/36 + 5/36 + 3/36 + 1/36 = ½.

3. (10%) Work out Problem 1-4.6 in Cooper and Mcgillem.

Horsepower	120V ac	240V ac	240V 3-p
0.1	900	400	0
0.5	200	500	100
1.0	100	200	600

   Assume that 10% of the 120V and 5% of the 240 V ac motors from the above Table are mismarked and evenly distributed over all 3 power levels, what is:

   a) Probability of randomly selecting a mismarked one:
      P("mismarked")=([10% of 1200] + [5% of 1100])/3000=7/120;
   b) Probability of selecting a mismarked one from those marked 240V ac :
      P("mismarked"| "240V ac")=P("mismarked" and "240V ac")/P("240V ac") = (55/3000)/(1100/3000)= 0.05;
   c) Probability of selecting one that is 0.5hp and is mismarked:
      P("mismarked" and "0.5 hp") = (0.1*200+0.05*500)/3000= 0.015.

4. (10%) Work out Problem 1-6.3 in Cooper and Mcgillem.

   Assume that three cards are picked randomly in succession from a standard, unbiased, and fair deck of 52 playing cards. Define the following three events: A={first card is a king}, B={second card is a king}, C={third card is a king}.

a) $P(A \cap \bar{B}) = P(A) * P(\bar{B} \mid A) = \frac{1}{13} * \frac{48}{51} = 0.0724$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{13} + \frac{1}{13} - \frac{1}{13} * \frac{3}{51} = 0.14932$

c) $P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A})P(\bar{B} \mid \bar{A}) = \frac{12}{13} + \frac{12}{13} - \frac{12}{13} * \frac{47}{51} = 0.99548$

d) $P(\bar{A} \cap \bar{B} \cap \bar{C}) = \frac{12}{13} * \frac{47}{51} * \frac{46}{50} = 0.7826$

e)
$$
\begin{aligned}
&P((A \cap B) \cup (\bar{B} \cap C)) = P(A \cap B) + P(\bar{B} \cap C) - P(A \cap B \cap \bar{B} \cap C) \\
&= P(A \cap B) + [P(\bar{B} \cap C \mid A)P(A) + P(\bar{B} \cap C \mid \bar{A})P(\bar{A})] - P(\phi) \\
&= \tfrac{1}{13} * \tfrac{3}{51} + [\tfrac{1}{13} * \tfrac{48}{51} * \tfrac{3}{50} + \tfrac{12}{13} * \tfrac{47}{51} * \tfrac{4}{50}] + 0 = 0.07692
\end{aligned}
$$

f)
$$
\begin{aligned}
&P(\bar{A} \cup B \cup C) = P(\bar{A}) + P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \text{ (using Venn Diagram)} \\
&= 1 - P(A) + P(A \cap B) + [P(A \cap C \mid B)P(B) + P(A \cap C \mid \bar{B})p(\bar{B})] - P(A \cap B \cap C) \\
&= 1 - \tfrac{1}{13} + \tfrac{1}{13} * \tfrac{3}{51} + [\tfrac{1}{13} * \tfrac{3}{51} * \tfrac{2}{50} + \tfrac{1}{13} * \tfrac{48}{51} * \tfrac{3}{50}] - \tfrac{1}{13} * \tfrac{3}{51} * \tfrac{2}{50} = 0.93195
\end{aligned}
$$

5. (10%) Work out Problem 1-7.1 in Cooper and Mcgillem.

Define the following events: T0=transmitted 0, T1=transmitted 1, R0=received 0, and R1=received 1; P(T0)=0.4, P(T1)=0.6, P(R1|T0)=0.08, P(R0|T1)=0.05.

a) $P(T0 \mid R0) = \dfrac{P(R0 \mid T0) * P(T0)}{P(R0 \mid T0) * P(T0) + P(R0 \mid T1) * P(T1)} = \dfrac{(1-0.08) * 0.4}{(1-0.08) * 0.4 + 0.05 * 0.6} = 0.9246$

b) $P(T1 \mid R1) = \dfrac{P(R1 \mid T1) * P(T1)}{P(R1 \mid T0) * P(T0) + P(R1 \mid T1) * P(T1)} = \dfrac{(1-0.05) * 0.6}{0.08 * 0.4 + (1-0.05) * 0.6} = 0.9468$

c) $P("Error") = P(R0 \mid T1) * P(T1) + P(R1 \mid T0) * P(T0) = 0.05 * 0.6 + 0.08 * 0.4 = 0.062$

6. (10%) Work out Problem 1-8.2 in Cooper and Mcgillem.

Three events A, B and C are independent, i.e. $P(A \cap B) = P(A) * P(B)$, $P(B \cap C) = P(B) * P(C)$, $P(A \cap C) = P(A) * P(C)$, $P(A \cap B \cap C) = P(A) * P(B) * P(C)$:

a)
$$
\begin{aligned}
&P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\
&= P(A) * P(B) + P(A) * P(C) - P(A) * P(B) * P(C) \\
&= P(A) * [P(B) + P(C) - P(B) * P(C)] = P(A) * P(B \cup C) \\
&\Rightarrow A \text{ and } B \cup C \text{ are independent}
\end{aligned}
$$

b)
$$
\begin{aligned}
&P(A \cap (B \cap C)) = P(A \cap B \cap C) = P(A) * P(B) * P(C) \\
&= P(A) * [P(B) * P(C)] = P(A) * P(B \cap C) \Rightarrow A \text{ and } B \cap C \text{ are independent}
\end{aligned}
$$

c)
$$
\begin{aligned}
& P(A\cap(B-C)) = P(A\cap(B\cap\bar{C})) = P(A\cap B\cap(S-C)) \\
& = P(A\cap B\cap S - A\cap B\cap C) = P(A\cap B) - P(A\cap B\cap C)\ [\because A\cap B\cap C \subset A\cap B] \\
& = P(A)*P(B) - P(A)*P(B)*P(C) = P(A)*P(B)*[1-P(C)] \\
& = P(A)*[P(B)*P(\bar{C})] = P(A)*[P(B-C)] \quad \Rightarrow \quad A \text{ and } B-C \text{ are independent}
\end{aligned}
$$

7. (10%) Work out Problem 1-9.2 in Cooper and Mcgillem.

a) They are independent with 2**4=16 possible outcomes: $GFCS, GFC\bar{S}, GF\bar{C}S, GF\bar{C}\bar{S}$, $G\bar{F}CS, G\bar{F}C\bar{S}, G\bar{F}\bar{C}S, G\bar{F}\bar{C}\bar{S}, \bar{G}FCS, \bar{G}FC\bar{S}, \bar{G}F\bar{C}S, \bar{G}F\bar{C}\bar{S}, \bar{G}\bar{F}CS, \bar{G}\bar{F}C\bar{S}, \bar{G}\bar{F}\bar{C}S, \bar{G}\bar{F}\bar{C}\bar{S}$.

b) $P(GFCS) = [1-P(\bar{G})]*[1-P(\bar{F})]*[1-P(\bar{C})]*[1-P(\bar{S})] = 0.95*0.9*0.97*0.88 = 0.7298$

c)
$$
\begin{aligned}
& P(\text{Only } \bar{F} \,|\, \text{"bad device"}) = P(G\bar{F}CS)/[1-P(GFCS)] \\
& = [0.95*0.1*0.97*0.88]/[1-0.7298] = 0.3001
\end{aligned}
$$

d) $P(\text{Both } \bar{F} \text{ and } \bar{C} \,|\, \text{"bad device"}) = P(\bar{F}\bar{C})/[1-P(GFCS)] = [0.1*0.03]/0.2702 = 0.0111$

8. (10%) Work out Problem 1-10.2 in Cooper and Mcgillem.

In playing an opponent of equal ability, which of the following is more likely:

a) Let A={winning 4 games out of 7}, B={winning 5 games out of 9}

$$P(A) = P_7(4) = \binom{7}{4}(\tfrac{1}{2})^4(\tfrac{1}{2})^{7-4} = \frac{7!}{4!3!}(\tfrac{1}{2})^7 = 0.2734$$

$$P(B) = P_9(5) = \binom{9}{5}(\tfrac{1}{2})^5(\tfrac{1}{2})^{9-5} = \frac{9!}{5!4!}(\tfrac{1}{2})^9 = 0.2461$$

So event A (4 out of 7) is more likely than event B (5 out of 9);

b) Let C={winning at least 4 out of 7}, D={winning at least 5 out of 9}

$$P(C) = P_7(4)+P_7(5)+P_7(6)+P_7(7) = (\frac{7!}{4!3!}+\frac{7!}{5!2!}+\frac{7!}{6!1!}+1)*(\tfrac{1}{2})^7 = 0.5$$

$$P(D) = P_9(5)+P_9(6)+P_9(7)+P_9(8)+P_9(9) = (\frac{9!}{5!4!}+\frac{9!}{6!3!}+\frac{9!}{7!2!}+\frac{9!}{8!1!}+1)*(\tfrac{1}{2})^9 = 0.5$$

So events C (at least 4 out of 7) and D (at least 5 out of 9)*are* equally likely.

9. (20%) Work out Problem 1-10.9 in Cooper and Mcgillem (MATLAB).

Simply follow the MATLAB code to obtain the experimental results:

a) Plot the accumulated winning after 50 repetitions (more plays, more loss);
b) Because the winning odds are less than 50% in any single spin of the wheel, and there is a finite limit of the amount than can be bet to recover the loss;
c) If we set the single spin winning percentage at 50%, there is a fair chance the bettor will win or lose in the long run.