Solution to Homework 6 (ECE3075 Summer 2004)

1. (10%) Work out Problem 6-1.1 in Cooper and McGillem.

$Y(t) = X(t) + bX(t-1) : E(Y(t)] = E[X(t)] = 0, \sigma_X^2 = R_X(0) = 5,$

$\sigma_Y^2 = E[X^2(t)] + 2bE[X(t)X(t-1)] + b^2 E[X^2(t-1)] = (1+b^2)R_X(0) + 2bR_X(1).$

(a) $d\sigma_Y^2 / db = 2bR_X(0) + 2R_X(1) = 0 \Rightarrow \sigma_Y^2$ is minimized at $b = -R_X(1)/R_X(0)$, with $R_X(1) = 5e^{-5}, R_X(0) = 5 \Rightarrow b = -e^{-5} = -0.00674$;

(b) $\sigma_Y^2 = (1+b^2)R_X(0) + 2bR_X(1) = R_X(0)[1-b^2] = 5[1-e^{-10}] \approx 5$;

(c) For $|b| \le 1, \sigma_Y^2$ is maximized when b is negative at b = -1, $\sigma_Y^2 = (1+b^2)R_X(0) + 2bR_X(1) = 2(1-e^{-5})R_X(0) \approx 10.$

2. (10%) Work out Problem 6-2.1 in Cooper and McGillem [Recall HW3-8].

$X(t) = \sum_{n=-\infty}^{\infty} A(t) f(t - nT - t_0)$, $A(t) = 0$ or $A(t) = 1$ equally likely, and t_0 is $U(0,T)$; we have $0 \le T_1 \le T/2$, with $f(t) = u(t) - u(t-T_1), \ni u(t)$ is a step function with a discontinuity at $t = 0$

a) For any given $t, X(t)$ is defined for a given t_0 for only one k with an interval from 0 to T (near $t-kT$), so $E[X(t)] = E[A(t)] * E[f(t-t_0-kT)] = \frac{1}{2} * \int_{t-kT-T_1}^{t-kT} \frac{1}{T} dt_0 = \frac{1}{2} * \frac{T_1}{T} = \frac{T_1}{2T}$, ;

therefore $E[X^2(t)] = E[A^2(t)] * E[f^2(t-t_0-kT)] = \frac{1}{2} * \int_{t-kT-T_1}^{t-kT} \frac{1}{T} dt_0 = \frac{T_1}{2T}$

b) $R_X(\tau) = E[X(t)X(t+\tau)] = E[A(t)A(t+\tau)] * E[f(t-t_0-kT) f(t+\tau-t_0-jT)]$

Now $E[A(t)A(t+\tau)] = \frac{1}{2}$ (if $j = k$, or the gap τ is small that they fall in the same pulse)

On the other hand, $E[A(t)A(t+\tau)] = \frac{1}{4}$ (if $j \ne k$, so $A(t)$=$A(t+\tau) = 1$ only 25% of the time);

For any τ, only one value of (j - k) contributes to the computation of the following:

For $\tau \ge 0$, $E[f(t-t_0-kT) f(t+\tau-t_0-jT)] = \int_{t+\tau-jT-T_1}^{t-kT} \frac{1}{T} dt_0 = \frac{T_1 - [\tau - (j-k)T]}{T}$,

with $\tau - (j-k)T \le T_1$, similar computation can be done for $\tau < 0$, therefore

$$R_X(\tau) = \frac{T_1}{2T} * [1 - \frac{|\tau|}{T_1}] \text{ (if } j = k \text{, or the gap } |\tau| \le T_1)$$

$$= \frac{T_1}{4T} * [1 - \frac{|\tau - (j-k)T|}{T_1}] \text{ (if } j \ne k \text{, or } |\tau - (j-k)T| \le T_1)$$

$$= 0 \text{ (elsewhere)}$$

3. (10%) Work out Problem 6-3.2 in Cooper and Mcgillem.

$X(t) = Y\cos(\omega_0 t + \theta), Y, \omega_0$ and θ are independent. Assume Y has a mean of 3 and a variance of 9, θ is uniform over [-π,π], and ω_0 is uniform over [-6,6].

(a) *X(t)* is stationary but not ergodic [see parts (b) and (c)];

$E[X(t)] = E[Y]*(E[\cos\omega_0 t]E[\cos\theta]\text{-}E[\sin\omega_0 t]E[\sin\theta])=0$

(b) $E[X^2(t)] = E[Y^2]*\{\tfrac{1}{2}(1+E[\cos 2\omega_0 t]E[\cos 2\theta]\text{-}E[\sin 2\omega_0 t]E[\sin 2\theta])\}=18*\tfrac{1}{2}=9$;

Using $\cos(\alpha+\beta)=[\cos\alpha\cos\beta-\sin\alpha\sin\beta]$, $\cos 2\alpha = 2\cos^2\alpha - 1$(Appendix A)

$$E[X(t)X(t+\tau)] = E[Y^2]*E\{[\cos(\omega_0 t+\theta)]*[\cos(\omega_0 t+\omega_0\tau)\cos\theta-\sin(\omega_0 t+\omega_0\tau)\sin\theta]\}$$

$$= E[Y^2]*E\{\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)\cos^2\theta+\sin(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)\sin^2\theta$$
$$-\sin(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)\cos\theta\sin\theta-\cos(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)\cos\theta\sin\theta\}$$

(c) $$=18*E\{[\tfrac{1}{2}\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)]+[\tfrac{1}{2}\sin(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)]-[\sin(2\omega_0 t+\omega_0\tau)\cos\theta\sin\theta]\}.$$

Using $E[\cos^2\theta]=E[\tfrac{1}{2}\cos(2\theta)+\tfrac{1}{2}]=\tfrac{1}{2}$, and $E[\sin^2\theta]=E[-\tfrac{1}{2}\cos(2\theta)+\tfrac{1}{2}]=\tfrac{1}{2}$,

so $R_X(\tau)=E[X(t)X(t+\tau)]=18*\{\tfrac{1}{2}[\tfrac{1}{2}\cos(\omega_0\tau)+\tfrac{1}{2}\cos(\omega_0\tau)]=9*E[\cos(\omega_0\tau)]=\dfrac{3}{2\tau}\sin(6\tau)$

Using $E[\cos\theta\sin\theta]=E[\tfrac{1}{2}\sin(2\theta)]=0$, $E[\cos(\omega_0 t)\sin(\omega_0 t)]=E[\tfrac{1}{2}\sin(2\omega_0 t)]=0$,

$E[\cos^2(\omega_0 t)]=E[\tfrac{1}{2}\cos(2\omega_0 t)+\tfrac{1}{2}]=\tfrac{1}{2}$, and $E[\sin^2(\omega_0 t)]=E[-\tfrac{1}{2}\cos(2\omega_0 t)+\tfrac{1}{2}]=\tfrac{1}{2}$

4. (10%) Work out Problem 6-4.2 in Cooper and Mcgillem [Recall HW3-9].

i	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)
0	0.19	3	0.83	6	-1.47	9	-0.31	12	0.57	15	-0.82	18	0.91
1	0.29	4	-0.01	7	-1.24	10	1.18	13	0.95	16	-0.25	19	-0.19
2	1.44	5	-1.23	8	-1.88	11	1.70	14	1.45	17	0.23	20	0.24

$\Rightarrow N=20, \bar{x}=0.0362$

Based on Eq. (6-15): $\hat{R}_X(k\Delta t)=\dfrac{1}{N-k+1}\sum_{i=1}^{N-k}x(i)x(i+k),\ k=0,1,2,3, \Delta t=0.01$

we have $\hat{R}_X(0)=1.002, \hat{R}_X(\Delta t)=0.581, \hat{R}_X(2\Delta t)=0.1669, \hat{R}_X(3\Delta t)=-0.0460$

Based on Eq. (6-16): $\hat{R}_X(k\Delta t)=\dfrac{1}{N+1}\sum_{i=1}^{N-k}x(i)x(i+k),\ k=0,1,2,3, \Delta t=0.01$

we have $\hat{R}_X(0)=1.002, \hat{R}_X(\Delta t)=0.553, \hat{R}_X(2\Delta t)=0.151, \hat{R}_X(3\Delta t)=-0.0387$

(a) $$\text{Var}[\hat{R}_X(k\Delta t)]\le\frac{2}{N}\sum_{k=-M}^{M}\hat{R}_X^2(k\Delta t)=0.1[(1.002)^2+2(0.581)^2+2(0.167)^2+2(-0.046)^2];$$

or $\text{Var}[\hat{R}_X(k\Delta t)]\le 0.1740$

(b) $\text{Var}[\hat{R}_X(k\Delta t)]\le 0.1[(1.002)^2+2(0.553)^2+2(0.151)^2+2(-0.0387)^2]=0.1664$.

5. (10%) Work out Problem 6-4.4 in Cooper and Mcgillem.

$R_X(t)=A[1-|t|/T]$ is the autocorrelation of a process that generates the data in Prob.6-4.1.

Let the regression function $y_t = R_X(t)=a+bt\ [a=A,\ b=\text{-}A/T,\ t=0,\tau,2\tau,3\tau,\ n=4]$, then

$$b=\frac{n\sum(i\tau*y_i)-\sum(i\tau)*\sum y_i}{n\sum(i\tau)^2-(\sum i\tau)^2}=\frac{n\sum(i*y_i)-(\sum i)*(\sum y_i)}{[4*14-36]*0.01}\ (\tau=0.01)$$

(a) $$=\frac{4(0+0.581+0.334-0.138)-6*(1.002+0.581+0.167-0.046)}{0.2}=\frac{3.108-10.224}{0.2}=-35.58;$$

$$a=\frac{\sum y_i-b\sum(i\tau)}{n}=\frac{(1.002+0.581+0.167-0.046)+35.58*6*0.01}{4}=0.9597,$$

therefore $A=a=0.9597, T=-A/b=0.027\Rightarrow R(t)=0.9597-35.58|t|$

Using Eq. (6-18) $\text{Var}[\hat{R}_X(k\Delta t)] \le \frac{1}{2T}\int_{-T}^{T} R_X^2(\tau)d\tau$

(b) $= \frac{1}{0.027}\int_0^{0.027}(0.9597 - 35.58\tau)^2 d\tau = \frac{0.000932}{0.027} = 0.0345$.

which is much less than the upper bound derived in HW6-4 above.

6. (10%) Work out Problem 6-5.4 in Cooper and McGillem.

$R_X(\tau) = 10e^{-2|\tau|} - 5e^{-4|\tau|}$

a) $E[X(t)] = E[X_1(t) - X_2(t)] = 0 \Rightarrow \text{Var}[X(t)] = E[X^2(t)] = \lim_{\tau\to 0} R_X(\tau)$
(difference of two zero-mean switching processes shown in Figure 6-6) ;
$E[X_1^2(t)] = \lim_{\tau\to 0} 10e^{-2|\tau|} = 10,\ E[X_2^2(t)] = \lim_{\tau\to 0} 5e^{-4|\tau|} = 5 \Rightarrow E[X^2(t)] = 5 = \sigma_X^2$,

b) $\frac{dR_X(\tau)}{d\tau} = \begin{cases} -(20e^{-2\tau} - 20e^{-4\tau}) & \tau > 0 \\ (20e^{-2\tau} - 20e^{-4\tau}) & \tau < 0 \end{cases} \Rightarrow$ continuous at $\tau = 0 \Rightarrow X(t)$ is differentiable.

7. (10%) Work out Problem 6-7.1 in Cooper and Mcgillem.

$X(t)$ and $Y(t)$ independent, $R_X(\tau) = 25e^{-10|\tau|}\cos(100\pi t), R_Y(\tau) = 16\frac{\sin(50\pi t)}{50\pi t}$, .

Define three processes: $Z(t) = X(t) + Y(t), W(t) = X(t) - Y(t), U(t) = X(t) * Y(t)$

a) $R_Z(\tau) = R_X(\tau) + R_Y(\tau)$;

b) $R_W(\tau) = R_X(\tau) - R_Y(\tau)$;

c) $R_{ZW}(\tau) = E[X(t)X(t+\tau)] - E[X(t)]E[Y(t+\tau)] + E[Y(t)]E[X(t+\tau)] - E[Y(t)Y(t+\tau)]$
$= R_X(\tau) - R_Y(\tau)$ [$Y(t)$ is zero mean, and $X(t)$ and $Y(t)$ are independent] ;

d) $R_U(\tau) = E[X(t)Y(t)X(t+\tau)Y(t+\tau)] = E[X(t)X(t+\tau)] * E[Y(t)Y(t+\tau)] = R_X(\tau) * R_Y(\tau)$

8. (10%) Work out Problem 6-8.3 in Cooper and Mcgillem (MATLAB).
Left as an exercise.

9. (10%) Work out Problem 6-8.8 in Cooper and Mcgillem.

$\tau = D\sin\theta / c \Rightarrow \theta = \sin^{-1}(\tau c / D)$ [c is the speed of light],

$d\theta = \frac{\partial\theta}{\partial\tau}d\tau + \frac{\partial\theta}{\partial D}dD = \frac{c}{D\cos\theta}d\tau - \frac{1}{D}\tan\theta dD$ (assume $d\tau$ and dD are independent),

$\sigma_\theta^2 = (\frac{c}{D\cos\theta})^2\sigma_\tau^2 + (\frac{1}{D}\tan\theta)^2\sigma_D^2 \le (0.001)^2 = 10^{-6}\,\text{rad}^2$,

$\sigma_\tau^2 \le (\frac{500\cos\theta}{3*10^8})^2[10^{-6} - (\frac{1}{500}\tan\theta)^2 * 10^{-4}]$, .

for $\theta = 0$ rad $\Rightarrow \sigma_\tau^2 \le 2.778*10^{-18}$, $\theta = 1.4$ rad $\Rightarrow \sigma_\tau^2 \le 7.917*10^{-20}$,

therefore the upperbound is $\sqrt{7.917*10^{-20}} = 2.81*10^{-10}$ or 0.281 nsec

10.(10%) Work out Problem 6-9.4 in Cooper and Mcgillem (MATLAB).

$Y(t) = a_0 X(t) + a_1 X(t-\Delta t) + a_2 X(t-2\Delta t) + a_3 X(t-3\Delta t)$,

$R_Y(\tau) = E[Y(t)Y(t+\tau)] = \sum_{i=0}^{3}\sum_{j=0}^{3} a_i a_j E[X(t-i\Delta t)X(t-j\Delta t+\tau)] = \sum_{i=0}^{3}\sum_{j=0}^{3} a_i a_j R_X((i-j)\Delta t+\tau)$.

$R_X(\tau) = 1-|\tau|/\Delta t \quad [\,|\tau| < \Delta t] \Rightarrow$ sum is a decayed trangular, spaced by Δt

(a) $a_0 = a_1 = a_2 = a_3 = 1$,

$R_Y(\tau) = \sum_{i=0}^{3}\sum_{j=0}^{3} R_X((i-j)\Delta t+\tau)$ [Autocorrelation peaked at $R_Y(0) = 4$];

(b) $a_0 = 4, a_1 = 3, a_2 = 2, a_3 = 1$,

$R_Y(\tau) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_i a_j R_X((i-j)\Delta t+\tau)$ [Autocorrelation peaked at $R_Y(0) = 30$]