Solution to Homework 7 (ECE3075 Summer 2004)

1. (10%) Work out Problem 7-1.1 in Cooper and McGillem.
   $X(t) = M\text{rect}[t/2T]$, with M being U(-6,18)..
   (a) $E[X(t)] = E[M]\text{rect}(t/2T) = 6\text{rect}(t/2T)$;
   (b) $X(f) = \int_{-T}^{T} Me^{-j2\pi ft}dt = M * \frac{-1}{\pi f}[\frac{e^{-j2\pi fT} - e^{j2\pi T}}{2j}] = 2TM\text{sinc}(2Tf)$;
   (c) $E[X(f)] = 2TE[M] * \text{sinc}(2Tf) = 12T\text{sinc}(2Tf)$;
   (d) $\lim_{T\to\infty} X(f) = \lim_{T\to\infty} 2TM * \text{sinc}(2Tf) = M * \lim_{T\to\infty} [\sin(2\pi Tf)/\pi f] = M\delta(f)$.

2. (10%) Work out Problem 7-2.1 in Cooper and McGillem [Recall HW3-8].
   a) $F^{-1}[8\sin(4\omega)/4\omega] = \text{rect}[t/8],\ F^{-1}[16\sin(8\omega)/8\omega] = \text{rect}[t/16]$
   $\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(4\omega)}{4\omega} * \frac{\sin(8\omega)}{8\omega} d\omega = \frac{2\pi}{8*16}\int_{-4}^{4} 1dt = \frac{\pi}{8}$;
   b) $\frac{1}{\omega^4 + 5\omega^2 + 4} = \frac{1}{(\omega^2+1)(\omega^2+4)}$
   $F^{-1}[1/(\omega^2+1)] = \frac{1}{2}e^{-|t|},\ F^{-1}[1/(\omega^2+4)] = \frac{1}{4}e^{-2|t|}$
   $\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(\omega^2+1)(\omega^2+4)} d\omega = \frac{2\pi}{8}\int_{-\infty}^{\infty} e^{-|t|}e^{-2|t|}dt = \frac{4\pi}{8}\int_{-\infty}^{\infty} e^{3t}dt = \frac{\pi}{6}$.

3. (10%) Work out Problem 7-3.1 in Cooper and Mcgillem.
   (a) Not even; (b) Yes; (c) Yes; (d) Might be negative; (e) Yes; (f) Not even.

4. (10%) Work out Problem 7-4.1 in Cooper and Mcgillem [Recall HW3-9].
   $S_X(\omega) = [16(\omega^2+36)/(\omega^4+13\omega^2+36)]$.
   (a) $S_X(s) = [16(-s^2+36)/(s^4-13s^2+36)]\ \ (\omega^2 = -s^2)$;
   (b) $S_X(s) = \frac{16(-s+6)(s+6)}{(s+2)(s-2)(s+3)(s-3)} \Rightarrow$ poles: $\pm 2, \pm 3$, zeroes: ± 6
   (c) $f = 1 \Rightarrow \omega = 2\pi \Rightarrow S_X(2\pi) = \frac{16[(2\pi)^2+36]}{(2\pi)^4+13(2\pi)^2+36} = 0.57314$;
   (d) $s' = s/100 \Rightarrow S_X(s') = \frac{160000(-s+600)(s+600)}{(s+200)(s-200)(s+300)(s-300)}$.

5. (10%) Work out Problem 7-5.4 in Cooper and Mcgillem.
   $S_X(\omega) = [(\omega^2+10)/(\omega^4+5\omega^2+4)] + 8\pi\delta(\omega) + 2\pi\delta(\omega-3) + 2\pi\delta(\omega+3)$.

$$E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\omega^2+10)}{(\omega^2+1)(\omega^2+4)} d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} [8\pi\delta(\omega) + 2\pi\delta(\omega-3) + 2\pi\delta(\omega+3)]d\omega$$
$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{(-s+\sqrt{10})(s+\sqrt{10})}{(s+1)(s-1)(s+2)(s-2)} ds + \frac{1}{2\pi}[8\pi + 2\pi + 2\pi]$$
$$= \frac{-s^2+10}{(s-1)(s^2-4)}\Big|_{s=-1} + \frac{-s^2+10}{(s^2-1)(s-2)}\Big|_{s=-2} + 6 = \frac{9}{6} - \frac{6}{12} + 6 = 7$$

6. (10%) Work out Problem 7-6.2 in Cooper and McGillem.

$R_X(\tau)=16e^{-5|\tau|}\cos(20\pi\tau)+8\cos(10\pi\tau)$.

a) $E[X^2(t)]=R_X(0)=16+8=24, \bar{X}=0 \Rightarrow \sigma_X^2=24$;

b)
$$S_X(\omega)=\int_{-\infty}^{\infty}16e^{-5|\tau|}\cos(20\pi\tau)e^{-j\omega\tau}d\tau+\int_{-\infty}^{\infty}8\cos(10\pi\tau)e^{-j\omega\tau}d\tau$$
$$=16[\frac{5}{25+(\omega-20\pi)^2}+\frac{5}{25+(\omega+20\pi)^2}]+8\pi[\delta(\omega-10\pi)+\delta(\omega+10\pi)];$$

c) $S_X(0)=16[\frac{5}{25+(0-20\pi)^2}+\frac{5}{25+(0+20\pi)^2}]=\frac{160}{25+400\pi^2}=0.0403$.

7. (10%) Work out Problem 7-7.1 in Cooper and Mcgillem.

$S_X(\omega)=9/(\omega^2+64)\Rightarrow S_X(0)=9/64$,

$E[X^2(t)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{9}{\omega^2+64}d\omega=\frac{1}{2\pi}*\frac{9}{8}\tan^{-1}(\omega/8)|_{-\infty}^{\infty}=9/16$.

a) $S_N(0)=9/64, E[N^2(t)]=2\omega_N S_N(0)=9/16=9\omega_N/32\Rightarrow\omega_N=2,$
$S_N(\omega)=9/64,\ |\omega|\le 4\pi; S_N(\omega)=0,\ |\omega|>4\pi$;

b) $R_X(\tau)=F^{-1}[9/(\omega^2+64)]=\frac{9}{16}e^{-8|\tau|}$;

c) $R_N(\tau)=F^{-1}[S_N(\omega)]=2*2*\frac{9}{64}[\frac{\sin(2*\pi*2*\tau)}{2*\pi*2*\tau}]=\frac{9}{16}\frac{\sin(4\pi\tau)}{4\pi\tau}$;

d) $R_X(0)=9/16=R_N(0)$, since $E[X^2]$=$E[N^2]$, two autocorrelations cover the same area

8. (10%) Work out Problem 7-8.1 in Cooper and Mcgillem.

$S_X(\omega)=16/(\omega^2+16), S_Y(\omega)=\omega^2/(\omega^2+16), X$ and Y independent,$E[X(t)]=0$.

(a) $U(t)=X(t)+Y(t), R_U(\tau)=R_X(\tau)+R_Y(\tau), S_U(\omega)=S_X(\omega)+S_Y(\omega)=1\ \forall\omega$;

(b) $R_{XY}(\tau)=E[X(t)Y(t+\tau)]=E[X(t)]*E[Y(t+\tau)]=0\Rightarrow S_{XY}(\omega)=0$;

(c) $R_{UX}(\tau)=E[X(t)\{X(t+\tau)+Y(t+\tau)\}]=R_X(\tau)+R_{XY}(\tau)=R_X(\tau)\Rightarrow S_{UX}(\omega)=16/(\omega^2+16)$.

9. (10%) Work out Problem 7-9.1 in Cooper and Mcgillem (MATLAB).
Left as an exercise.

10. (10%) Work out Problem 7-11.2 in Cooper and Mcgillem.

(a) $d(s)=(s+10+100j)(s+10-100j)(s-10+100j)(s-10-100j)=(s^2+20s+10100)(s^2-20s+10100)$

$\Rightarrow d(\omega)=\omega^4-19800\omega^2+102010000\Rightarrow$ maximizing $S(\omega)$= minimizing $d(\omega)$

$\partial d(\omega^2)/\partial(\omega^2)=0\Rightarrow\omega_0=\sqrt{19800/2}=99.5\Rightarrow d(\omega_0)=(1.0201-0.99^2)*10^8=4*10^6$,

or half power point at ω_1 with $d(\omega_1)=2d(\omega_0)=8*10^6\Rightarrow\omega_1=\sqrt{9900\pm1414}$

or $\omega_1=91.12$ and $106.37\Rightarrow$ the bandwidth BW=$2\Delta\omega=2|\omega_1-\omega_0|$, or

BW=16.76 or 13.7 rad $\Rightarrow 2.67Hz$ or $2.18Hz$ (two-sided, two possibilities)

(b) 1% power point at ω_2 with $d(\omega_2)=100d(\omega_0)=4*10^8$, or

$\omega_1=\sqrt{9900+\sqrt{9900^2+2.98*10^8}}=172.63\Rightarrow$ the bandwidth BW=$2|\omega_2-\omega_0|$ or $23.28Hz$