SOLUTIONS to Quiz #1

Problem 1 (20%)

Define $\xi = (X - \mu_X)/\sigma_X$ and $\eta = (Y - \mu_Y)/\sigma_Y$, then we have the pdf

$$f(\xi,\eta) = \frac{1}{(\sqrt{2\pi})^2\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-\rho^2)}(\xi^2+\eta^2-2\rho\xi\eta)]$$

(a) $$f(\eta) = \frac{1}{(\sqrt{2\pi})} * \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2(1-\rho^2)}[(\xi-\rho\eta)]^2+\eta^2-\rho^2\eta^2)]d\xi$$

$$= \frac{1}{(\sqrt{2\pi})}\exp[-\frac{\eta^2}{2}] \Rightarrow \eta \text{ is a Gaussian random variable}$$

so the pdf of X is $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp[-\frac{(x-\mu_X)^2}{2\sigma_X^2}]$;

(b) $$f(\xi \mid \eta) = f(\xi,\eta)/f(\eta) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{-\frac{1}{2(1-\rho^2)}[(\xi-\rho\eta)]^2\}$$

or the pdf of X given $Y = y$ is Gaussian with the following pdf:

$$f(x \mid y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_X}\exp\{-\frac{1}{2(1-\rho^2)\sigma_X^2}[(x-\mu_X)-\frac{\rho\sigma_X}{\sigma_Y}(y-\mu_Y)]^2\};$$

(c) The conditional variance of X given $Y = y$ is $(1-\rho^2)\sigma_X^2$, and the conditional mean of X given y is $\mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y-\mu_Y)$;

(d) Since $(1-\rho^2)\sigma_X^2 \le \sigma_X^2$, knowing $Y = y$ will reduce the variance of X given y.

Problem 2 (20%)

$\mu_X = \mu_Y = \mu_Z = 0, \sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = 1$, and $\rho_{XY} = 0, \rho_{XZ} = 1/2, \rho_{YZ} = -1/2$;

$W = X + Y + Z$, and $U = Y - Z \Rightarrow \mu_W = \mu_U = 0$.

(a) $\sigma_W^2 = E[(W-\mu_W)^2] = E[X^2+Y^2+Z^2+2XY+2YZ+2XZ]$

$= \sigma_X^2+\sigma_Y^2+\sigma_Z^2+2\rho_{XY}\sigma_X\sigma_Y+2\rho_{YZ}\sigma_Y\sigma_Z+2\rho_{XZ}\sigma_X\sigma_Z = 3$;

(b) $\rho_{WX} = E[(W-\mu_W)(X-\mu_X)]/\sigma_W\sigma_X = E[X^2+XY+XZ]/\sigma_W\sigma_X$

$= [\sigma_X^2+\rho_{XY}\sigma_X\sigma_Y+\rho_{XZ}\sigma_X\sigma_Z]/\sigma_W\sigma_X = \sqrt{3}/2$;

(c) $\sigma_U^2 = E[(U-\mu_U)^2] = E[Y^2-2YZ+Z^2] = 3$

$\rho_{WU} = E[(W-\mu_W)(U-\mu_U)]/\sigma_W\sigma_U = E[XY-XZ+Y^2-Z^2]/\sigma_W\sigma_U$

$= (\rho_{XY}-\rho_{XZ}+\sigma_Y^2-\sigma_Z^2)/\sigma_W\sigma_U = (-\tfrac{1}{2})/3 = -1/6$.

Problem 3 (20%)

$X(t) = \cos(100t+\Theta)$ and $Y(t) = \cos(100t+\Psi)$, with both Θ and Ψ being independent, and uniformly distributed from 0 to 2π.

(a) $Z(t) = X(t) + Y(t) = \cos(100t + \Theta) + \cos(100t + \Psi) = A\cos(100t + \Phi);$ with

$A = 2\cos(\frac{\Theta - \Psi}{2})$ and $\Phi = \frac{\Theta + \Psi}{2}$ $[\cos X + \cos Y = 2\cos(\frac{X - Y}{2})\cos(\frac{X + Y}{2})];$

(b) Φ has a triangular distribution between 0 and 2π with a pdf

$$f_\Phi(\phi) = \begin{cases} \phi / \pi^2 & 0 \le \phi \le \pi \\ (2\pi - \phi)/\pi^2 & \pi \le \phi \le 2\pi; \\ 0 & \text{otherwise} \end{cases}$$

so $P(\phi \le \pi/3) = \int_0^{\pi/3} f_\Phi(\phi)d\phi = \phi^2 / 2\pi^2 |_0^{\pi/3} = 1/18;$

(c) Define $\alpha = (\Theta - \Psi)/2,$ then α has a triangular distribution between -π and π with

$$f(\alpha) = \begin{cases} (\phi + \pi)/\pi^2 & -\pi \le \phi \le 0 \\ (\pi - \phi)/\pi^2 & 0 \le \phi \le \pi \ ; \\ 0 & \text{otherwise} \end{cases}$$

(d) $P(A > 1) = P(\cos\alpha > 1/2) = P(|\alpha| < \pi/3)$

$= \int_{-\pi/3}^{\pi/3} f(\alpha)d\alpha = -2(\pi - \phi)^2 / 2\pi^2 |_0^{\pi/3} = -(4/9) + 1 = 5/9.$

Problem 4 (20%)

(a) The total number of possible combinations for drawing two cards is $J = {}_{52}C_2 = (52*51)/2 = 1326$. There are 4 Aces and 16 cards (10, Jack, Queen and King, four each) to make up a total of $K = 64$ combinations of blackjack hands. So the probability of $K/J = 64/1326 = 0.04826546;$

(b) Let A be the event that the delaer get a blackjack hand and B be the event that the player get a blackjack hand, then $P(A \text{ and } B) = P(A|B)P(B) = 0.01773;$ [since $P(A|B) = [(4-1)*(16-1)]/ {}_{50}\text{C}_2 = 45/1225 = 0.036735,\ P(B) = 0.048265$]

(c) The total number of possible combinations for drawing five cards is $M = {}_{52}\text{C}_5 = (52*51*50*49*48)/(5*4*3*2) = 2598960$. To get a full house hand, there are 13*12 types of three-of-a-kind and pair combinations, so the total number is $N = 13*12*{}_4C_3*{}_4C_2 = 3744$. So $P(\text{"}Full\ House\text{"}) = N/M = 0.001441;$

(d) Continuing from parts (c), to get a four-of-a-kind hand, there are L=13*48=624 combinations, so $P(\text{"four-of-a-kind"}) = L/M = 0.0002401.$

Problem 5 (10%)

(a) *P*(R appears)=*P(*R|R*)P(*R*)+P(*R|E*)P(*E*)+P(*R|T*)P(*T*)*
=0.96*0.0484+0.02*0.1031+0.02*0.0796=0.0501;

(b) *P*(Error | R appears)=[*P(*R|E*)P(*E*)+P(*R|T*)P(*T*)]/ P*(R appears)
=[0.02*0.1031+0.02*0.0796]/0.0501=0.0729.

Problem 6 (20%)

(a) $P_n(X=k) = {}_nC_k p^k (1-p)^{n-k}$

$$E[X] = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} k p^k (1-p)^{n-k}$$

$$= np * \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np * \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} p^m (1-p)^{m-l} = np$$

[with a change of variable of $m = n-1$ and $l = k$ - 1]

$$E[X^2 - X] = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (k^2 - k) p^k (1-p)^{n-k}$$

$$= n(n-1)p^2 * \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k}$$

$$= n(n-1)p^2 * \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} p^m (1-p)^{m-l} = n(n-1)p^2$$

$\Rightarrow$ $\text{Var}[X] = E[X^2 - X] + E[X] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$;

(b) $\phi(u) = (1-p+pe^{ju})^n$ and $E[X^m] = j^{-m} * d^m\phi(u)/du^m \mid_{u=0}$.

Since $\dfrac{d\phi(u)}{du} = npje^{ju}(1-p+pe^{ju})^{n-1}$ and

$$\frac{d^2\phi(u)}{du^2} = npj^2e^{ju}(1-p+pe^{ju})^{n-1} + n(n-1)pje^{ju}(1-p+pe^{ju})^{n-2} * pje^{ju}$$

so $E[X] = j^{-1}d\phi(u)/du \mid_{u=0} = np$ [same as in part (a)],

$E[X^2] = j^{-2}d^2\phi(u)/du^2 \mid_{u=0} = np + n(n-1)p^2 = (np)^2 + np(1-p)$,

therefore the variance Var(X)=$E[X^2] - (E[X])^2 = np(1-p)$ [same as in part (a)];

(c) (*Extra Credit* +10%) For $n = 100{,}000$, and $p = 0.5$, we have μ=$E[X] = 50{,}000$, and σ=$\sqrt{\text{Var}(X)} = \sqrt{25000} = 158.11$. From the given table, we know that with a Gaussian approximation $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9974$, or $P(49525.67 < X < 50474.33) = 0.9974$ if the voters do not favor either candidate. The actual value of $X = 49{,}000$ shows that the probability that Anderson will win is almost zero, and therefore his opponent is likely to win.