SOLUTIONS to Quiz #2

Problem 1 (20%)

The joint pdf $f(x,y) = f(x) * f(y) = 1/36,\ 1 \le x \le 6, 1 \le y \le 6$,

(a) $P(X \le 3, Y > 3) = \sum_{x=1}^{3}\sum_{y=4}^{6} \frac{1}{36} = \frac{1}{36} * 3 * 3 = \frac{1}{4}$;

(b) $E[XY] = \sum_{x=1}^{6}\sum_{y=1}^{6} \frac{xy}{36} = \frac{1}{36} * 21 * 21 = \frac{49}{4} = 12.25 = E[X] * E[Y] = \frac{21}{6} * \frac{21}{6}$;

(c) $E[\frac{X}{Y}] = \sum_{x=1}^{6} \frac{x}{6} \sum_{y=1}^{6} \frac{1}{6y} = \frac{21}{6} * \frac{1}{6}(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}) = \frac{343}{240} = 1.4292 = E[X] * E[\frac{1}{Y}]$.

Problem 2 (20%)

$f(x,y) = K \exp[-(x^2 + xy + y^2)]$ [X and Y are jointly Gaussian, check Eq. (3-26)].

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} K \exp[-(x^2 + xy + \frac{1}{4}y^2)]dx * [e^{-\frac{3}{4}y^2}]\}dy$

$$= K\sqrt{\pi}\int_{-\infty}^{\infty} e^{-\frac{3}{4}y^2} dy = K\sqrt{\pi} * \sqrt{2\pi * (2/3)} = 1 \Rightarrow K = \frac{\sqrt{3}}{2\pi};$$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} K \exp[-(x^2 + xy + \frac{1}{4}y^2)]dx * [e^{-\frac{3}{4}y^2}]$

$$= K\sqrt{\pi} * e^{-\frac{3}{4}y^2} = \frac{1}{\sqrt{2\pi * (2/3)}} e^{-\frac{3}{4}y^2} \text{ (Normal r.v. with } \mu_Y = 0, \sigma_Y^2 = \frac{2}{3}\text{)},$$

;

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{K \exp[-(x^2 + xy + \frac{1}{4}y^2)] * e^{-\frac{3}{4}y^2}}{K\sqrt{\pi} * e^{-\frac{3}{4}y^2}}$$

$$= \frac{1}{\sqrt{\pi}} \exp[-(x + \frac{1}{2}y)^2] \text{ (Normal r.v. with } \mu_{X|y} = -\frac{1}{2}y, \sigma_{X|y}^2 = \frac{1}{2}\text{)}$$

(c) The conditional mean is $\mu_{X|y} = -y/2$;

(d) $\frac{\partial f(x \mid y)}{\partial x} = 0 \Rightarrow \hat{x} = \mu_{X|y} = -y/2$, the most likely estimate is the conditional mean .

Problem 3 (20%)

(a) $f_X(x) = e^{-x}u(x), \phi_X(u) = \int_0^{\infty} e^{-x} * e^{jux}dx = 1/[1 - ju]$,

$f_Y(y) = 3e^{-3y}u(y), \phi_Y(u) = \int_0^{\infty} 3e^{-3y} * e^{juy}dx = 3/[3 - ju]$;

Note we have the inverse relationship $\frac{1}{2\pi}\int_{-\infty}^{\infty} a/[a - ju] * e^{-jux}du = ae^{-ax}u(x)$

(b) $Z = X + Y \Rightarrow \phi_Z(u) = \int_0^{\infty}\int_0^{\infty} e^{-x} * 3e^{-3y} * e^{ju(x+y)}dxdy$

$$= \int_0^{\infty} e^{-x}e^{jux}dx * \int_0^{\infty} 3e^{-3y}e^{juy}dy = \phi_X(u) * \phi_Y(u) = \frac{3}{(1 - ju)(3 - ju)};$$

$$(c)\ f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_Z(u)*e^{-juz}du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{3}{(1-ju)(3-ju)}e^{-juz}du$$

$$= \frac{3}{2}*\frac{1}{2\pi}\int_{-\infty}^{\infty}[\frac{1}{(1-ju)} - \frac{1}{(3-ju)}]e^{-juz}du$$

$$= \frac{3}{2}[e^{-z} - e^{-3z}]u(z) \text{ [using inverse Fourier transform obtained in part (a)]}$$

Note that $\int_{-\infty}^{\infty} f_Z(z)dz = \frac{3}{2}\{-[0-1] + \frac{1}{3}[0-1]\} = 1 \Rightarrow f_Z(z)$ is a valid density

Problem 4 (20%)

$$\hat{\hat{X}} = \frac{1}{n}\sum_{i=1}^{n} X_i, S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\hat{X}})^2, E[X_i] = \mu, E[(X_i - \mu)^2] = \sigma^2.$$

$$(a)\ \text{Var}[\hat{\hat{X}}] = E[(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu)^2] = \frac{1}{n^2}\{E[\sum_{i=1}^{n}(X_i - \mu)^2] + E[\sum_{i=1}^{n}\sum_{j\neq i}^{n}(X_i - \mu)(X_j - \mu)]\}$$

$$= \frac{1}{n^2}*\{n\sigma^2 + 0\} = \frac{\sigma^2}{n}$$

$$(b)\ E[S_2^2] = \frac{1}{n}\sum_{i=1}^{n}E[(X_i - \hat{\hat{X}})^2] = \frac{1}{n}\sum_{i=1}^{n}E[(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j)^2]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\{E[X_i^2] - \frac{2}{n}E[X_i^2] - \frac{2}{n}\sum_{j\neq i}^{n}E[X_iX_j] + \frac{1}{n^2}\sum_{j=1}^{n}E[X_j^2] + \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k\neq j}^{n}E[X_jX_k]\}.$$

$$= \frac{1}{n}\sum_{i=1}^{n}\{\frac{(n-1)(\sigma^2+\mu^2)}{n} - \frac{2(n-1)}{n}\mu^2 + \frac{n-1}{n}\mu^2\} = \frac{n-1}{n}\sigma^2$$

Problem 5 (10%)

Let X be the random variable representing the ratio of property values between the years of 2000 and 1995. From random sampling, we have n=50, $\bar{x} = 1.534$, and $s_2 = 0.91$. We want to know if the true population mean $\mu_X = 1.0$ has changed.

(a) Null hypothesis $H_0 : \mu_X = 1.0$, alternative hypothesis $H_0 : \mu_X \neq 1.0$, we need to perform a two-sided test, we reject H_0 is a test statistic (to be formulated next) exceeds or falls below some threshold;

(b) Define a test statistic $z = \frac{\bar{x} - \mu_X}{\sigma_{\bar{X}}} = \frac{\bar{x} - \mu_X}{\sigma_X / \sqrt{n}} \approx \frac{\bar{x} - \mu_X}{s_2 / \sqrt{n}}$. We can assume the random variable z is approximately standardized Gaussian (zero mean and unit variance) because the sample size n is large enough (n > 30);

(c) At a significance level of $\alpha = 0.01$, we have $z_{\alpha/2} = 2.33$, since from the random samples, $z = 4.15 > z_{\alpha/2} = 2.33$, we reject the null hypothesis $H_0 : \mu_X = 1.0$;

(d) We can tell the City Council that if the assumption of housing price does not change in five years is indeed true, then based on our samples the probability of making the decision of rejecting the above hypothesis is less than 1%.

Problem 6 (20%)

(a) $Y(t) = X(t) + bX(t-\Delta t)$

$\Rightarrow E[Y(t)] = E[X(t)] + bE[X(t-\Delta t)] = (1+b)E[X(t)]$;

[Note: $E[X(t-\Delta t)] = E[X(t)]$, because $X(t)$ is wide sense stationary]

(b) $E[Y(t_1)Y(t_2)] = E[\{X(t_1)+bX(t_1-\Delta t)\}\{X(t_2)+bX(t_2-\Delta t)\}]$

$= E[X(t_1)X(t_2)] + bE[X(t_1)X(t_2-\Delta t)] + bE[X(t_1-\Delta t)X(t_2)] + b^2E[X(t_1-\Delta t)X(t_2-\Delta t)]$;

let $R_X(t_1,t_2) = E[X(t_1)X(t_2)]$, $X(t)$ is wide sense stationary $\Rightarrow R_X(t_1,t_2) = R_X(t_2-t_1) = R_X(t_1-t_2) \forall t_1, t_2$,

therefore $R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = R_X(\tau) + bR_X(\tau-\Delta t) + bR_X(\tau+\Delta t) + b^2R_X(\tau)$ $[\tau \triangleq t_2 - t_1]$

(c) From part (a) we know that $E[Y(t)]$ is independent of t just like $E[X(t)]$,

and $R_Y(t_1,t_2) = R_Y(t_2-t_1) = R_Y(\tau)$ just like $R_X(t_1,t_2)$,

$\Rightarrow$ we can concluse that $Y(t)$ is also wide sense stationary .