[9] V. V. Veeravalli, T. Basar, and H. V. Poor, "Decentralized sequential detection with a fusion center performing the sequential test," *IEEE Trans. Infor. Theory*, vol. 39, pp. 433–442, Mar. 1993.
[10] D. N. Jayasimha, S. A. Iyengar, and R. L. Kashyap, "Information integration and synchronization in distributed sensor networks," *IEEE Trans. Syst., Man, Cybern.*, vol. 21, pp. 1032–1043, Sep./Oct. 1991.
[11] A. Pete, K. R. Pattipati, and D. L. Kleinman, "Distributed detection in teams with partial information: A normative-descriptive model," *IEEE Trans. Syst., Man, Cybern.*, vol. 23, July/Aug. 1993.
[12] Z. B. Tang, K. R. Pattipati, and D. L. Kleinman, "A distributed M-ary hypothesis testing problem with correlated observations," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1042–1046, July 1992.
[13] E. Drakopoulos and C. C. Lee, "Optimum multisensor fusion of correlated local decisions," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 27, pp. 593–605, Jul. 1991.
[14] M. Kam, Q. Zhu, and W. S. Gray, "Optimal data fusion of correlated local decisions in multiple sensor detection systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, pp. 916–920, Jul. 1992.
[15] H. L. Van Trees, *Detection, Estimation, and Modulation Theory*, vol. I. New York: Wiley, 1968.
[16] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 2nd ed. New York: McGraw-Hill, 1984.
[17] F. L. Lewis, *Optimal Control*. New York: Wiley, 1986.
[18] A. Pete, K. R. Pattipati, and D. L. Kleinman, "Tasks and organizations: A signal detection model of organizational decisionmaking," to appear in *Int. J. Intell. Syst. Acc. Fin. Manag.* , vol. 2, pp. 289–303, Dec. 1993.

Nondifferentiability of the Steady-State Function in Discrete Event Dynamic Systems

A. Shapiro and Y. Wardi

***Abstract*—This paper suggests that expected-value performance functions in discrete-event dynamic systems can be nondifferentiable at dense sets of points in the parameter space, when the sample performance functions are convex and the distributions of events' times contain atoms. A general result is first proved for regenerative processes and then applied to simple queueing examples where nondifferentiability at dense sets is established.**

I. Introduction

Expected-value performance functions in discrete-event dynamic systems (DEDS) like queueing networks often lack closed-form expressions and hence are evaluated by Monte Carlo simulation. Estimation of their gradients can be done by sample path techniques like infinitesimal perturbation analysis [3], [4] or likelihood ratios/score functions [10]. This assumes, of course, that the above gradients do exist.

The question of whether gradients of such functions exist cannot always be easily answered. After all, corners of graphs produced by simulation can be attributed to the randomness involved. It is well known in the literature on perturbation analysis that the sample performance functions often are only piecewise differentiable, but their corners are not carried over to the steady state, where the expected performance functions are smooth (see [13], and [3], [4] and the references therein).

Until recently, no questions had been raised about the differentiability of steady-state (expected-value) performance functions. In fact, differentiability has been proved for various such functions in queueing networks under the assumption that the distributions of certain primitives like service times and interarrival times have densities [15], [6]. If some of these distributions contain atoms, the situation may be quite different. In that case, the expected-value function can be nondifferentiable at a set of points which forms a dense subset of the parameter space. This was first conjectured in [16] and subsequently proved for a particular system in [2]. The above system was quite simple and the expected-value function had a closed-form expression, and therefore the analysis could not be extended to more general networks. Nondifferentiability by graphical means was exhibited in [14] for queueing networks and for continuous flow models.

The main argument in [16] runs as follows. If some of the primitive's distributions contain atoms then, for every θ in a dense subset of the parameter space, with some positive probability, the sample performance functions have a corner, i.e., different one-sided derivatives. Moreover, these corners are carried over to the steady state and therefore, the expected-value performance function, $F(\theta)$, has an additive component that is not differentiable at θ. For $F(\cdot)$ to have a derivative at θ it must have another additive component that cancels the nondifferentiability. This, however, is not likely to happen.

The above argument is clearly heuristic and does not comprise a proof, and the paper [16] concludes with a general conjecture. The present paper proves the conjecture for a class of DEDS under the assumptions of convex (or, more generally, subdifferentiable) sample performance functions and various properties concerning regeneration. As in [16], a key condition is that, with some positive probability, the sample performance functions over a regeneration cycle are not differentiable at a particular point. Arguments from the theory of convex analysis then show that the nondifferentiability is carried over to the steady state. Finally, examples will demonstrate that, although a particular sample performance function typically is nondifferentiable only at a finite set of points, the expected-value function will get the nondifferentiabilities from infinitely many sample performance functions and, hence, can be nondifferentiable at a dense set of points.

It will become apparent from the analysis that the phenomenon of nondifferentiability can be quite general. The implications are, first, that the question of "gradient estimation" may have to be phrased in terms of subgradients (cf. [14]), and second, that sample path second-order optimization algorithms, based on approximations of the Hessians, may have to be ruled out.

Section II presents the general nondifferentiability result, Section III applies it to some examples, and Section IV concludes the paper.

II. Basic Results

Consider a function $F: \mathbb{R}^m \to \mathbb{R}$. It is said that F is directionally differentiable at a point $\theta \in \mathbb{R}^m$ if the limit

$$F'(\theta, d) = \lim_{t \to 0^+} \frac{F(\theta + td) - F(\theta)}{t}$$

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exists for every $d \in \mathbb{R}^m$. Suppose that, in addition, the directional derivative $F'(\theta, d)$ is convex in d. In this case, we say that the function F is subdifferentiable at θ (such functions were called locally convex in [5]). It follows then that there exists a convex, compact set $\partial F(\theta) \subset \mathbb{R}^m$ such that

$$F'(\theta, d) = \max_{z \in \partial F(\theta)} z^T d. \tag{2.1}$$

We will be particularly interested in real-valued, convex functions defined on a convex, open set $D \subset \mathbb{R}^m$. Such a function $F(\theta)$ is locally Lipschitz and subdifferentiable at every point $\theta \in D$, and the corresponding convex compact set $\partial F(\theta)$ is called the subdifferential of F at θ (see [8] for details). A larger class of subdifferentiable functions is given by composite functions $F(\theta) = G(A(\theta))$ of a convex function $G: \mathbb{R}^k \to \mathbb{R}$ and a differentiable mapping $A: \mathbb{R}^m \to \mathbb{R}^k$. Note that a subdifferentiable function F is differentiable at a point θ if and only if $\partial F(\theta) = \{z\}$ is a singleton, i.e., $\partial F(\theta)$ consists of one element z. In such case $F'(\theta, d)$ is linear in d and $z = \nabla F(\theta)$.

Now suppose that $F(\theta)$ has the form

$$F(\theta) = \mathbb{E}\{f(\theta)\} = \int_\Omega f(\theta, \omega) P(d\omega)$$

where $f(\theta)$ is a measurable random function (whose realization is denoted by $f(\theta, \omega)$ as in the above integral) defined on a probability space $(\Omega, \mathcal{F}, P)$. Let the domain of f be a convex open set $D \subset \mathbb{R}^m$ and suppose that, for every $\theta \in D$, $\mathbb{E}\{|f(\theta)|\} < \infty$. We now give sufficient conditions for directional differentiability of F at a point $\theta_0 \in D$ (cf. [10]).

Assumption 2.1: There exists a positive-valued random variable $K = K(\omega)$ such that $\mathbb{E}\{K\}$ is finite and

$$|f(\theta_1, \omega) - f(\theta_2, \omega)| \le K(\omega) \|\theta_1 - \theta_2\| \tag{2.2}$$

for almost all $\omega \in \Omega$ and for all $\theta_1, \theta_2 \in D$.

Assumption 2.2: With probability one (w.p.1) the function $f(\theta)$ is directionally differentiable at θ_0.

Proposition 2.1: Suppose that either i) Assumptions 2.1 and 2.2 hold, or ii) the function $f(\theta)$ is convex w.p.1. Then the expected-value function $F(\theta)$ is directionally differentiable at $\theta_0 \in D$ and

$$F'(\theta_0, d) = \mathbb{E}\{f'(\theta_0, d)\}. \tag{2.3}$$

Proof: Under Assumptions 2.1 and 2.2, formula (2.3) follows easily from the Lebesgue Dominated Convergence Theorem (see, e.g., [10, p. 307]). In the convex case, formula (2.3) is implied by the Monotone Convergence Theorem. Indeed, consider $\theta_0 \in D$, $d \in \mathbb{R}^m$, and a monotone-decreasing sequence $t_n \to 0^+$. It follows then from convexity of $f(\cdot, \omega)$, that the sequence $\psi_n = \psi_n(\omega) = t_n^{-1}[f(\theta_0 + t_n d, \omega) - f(\theta_0, \omega)]$ is monotone decreasing for almost every $\omega \in \Omega$, and that $\psi_n \to f'(\theta_0, d)$ w.p.1. Therefore by the Monotone Convergence Theorem

$$\lim_{n\to\infty} \mathbb{E}\{\psi_n\} = \mathbb{E}\Big\{\lim_{n\to\infty} \psi_n\Big\} = \mathbb{E}\{f'(\theta_0, d)\}.$$

Since the expected-value function $F(\theta) = \mathbb{E}\{f(\theta)\}$ is then convex, we have that the limit $\lim_{n\to\infty} \mathbb{E}\{\psi_n\}$ is finite and is equal to $F'(\theta_0, d)$. This shows that, in the convex case, if $\mathbb{E}\{|f(\theta)|\} < \infty$ for all $\theta \in D$, then the interchangeability formula (2.3) holds.

Suppose now that, in addition to the assumptions of Proposition 2.1, the function f is subdifferentiable at the point θ_0 w.p.1. Of course, in the convex case, this subdifferentiability follows from convexity of f. Then formula (2.3) implies that $F'(\theta, d)$ is convex in d and hence F is also subdifferentiable at θ_0. Moreover

$$\partial F(\theta_0) = \mathbb{E}\{\partial f(\theta_0)\} = \int_\Omega \partial f(\theta_0, \omega) P(d\omega) \tag{2.4}$$

where the integral of the multivalued mapping $\omega \to \partial f(\theta_0, \omega)$ in (2.4) is a subset of $\mathbb{R}^m$ whose points are the integrals of integrable selections of this multivalued mapping. For a detailed discussion of interchangeability of the subdifferential and integral operators (for convex functions) and relevant references see [5, chapter 8] and [9].

An important consequence of formula (2.4) is given in the following corollary.

Corollary 2.1: Suppose that the assumptions of Proposition 2.1 hold and that $f(\theta)$ is subdifferentiable at θ_0 w.p.1. Then, F is differentiable at the point θ_0 if and only if the subdifferential $\partial f(\theta_0)$ is a singleton w.p.1.

We next consider the case where $F(\theta)$ is the expected-value (mean) steady-state performance function of a DEDS and establish a condition under which nondifferentiabilities in the sample performance functions are carried over to the steady state. Consider a DEDS whose performance depends on a parameter $\theta \in \Theta$, with Θ being a convex, open subset of $\mathbb{R}^m$. Let $g_n(\theta)$, $n = 1, 2, \cdots$, be the sample performance functions. We assume that $g_n(\theta)$ are real-valued and convex functions of θ and that, for any fixed $\theta \in \Theta$, the processes $g_n(\theta)$ and $\partial g_n(\theta)$ are regenerative with regeneration cycles having finite first-order moments. In the examples below, $g_n(\theta)$ is the nth delay (or waiting) time at a queue in a stable queueing network, where both processes $g_n(\theta)$ and $\partial g_n(\theta)$ have the same regenerative cycles. In that case the corresponding length $\beta = \beta(\theta)$ of the first regenerative cycle represents the number of customers served in the first busy period.

Define $f_N(\theta) := N^{-1} \sum_{n=1}^N g_n(\theta)$ and $F_N(\theta) := \mathbb{E}\{f_N(\theta)\}$. It is well known in the theory of regenerative processes (e.g., [1], [17]) that as $N \to \infty$, $f_N(\theta)$ converge pointwise (i.e., for any fixed $\theta \in \Theta$) w.p.1 to the mean steady-state $F(\theta)$, and that

$$\lim_{N\to\infty} F_N(\theta) = F(\theta). \tag{2.5}$$

It follows from convexity of $g_n(\theta)$ that, the functions $f_N(\theta)$, and hence $F_N(\theta)$ and the limiting function $F(\theta)$, are also convex. Moreover, by convexity, it follows from the pointwise convergence (2.5) that

$$\limsup_{N\to\infty} F'_N(\theta, d) \le F'(\theta, d) \tag{2.6}$$

[8]. For two sets $U, V \subset \mathbb{R}^m$ we use the notation $U \subset_a V$ to mean that U is contained, up to an additive constant, in V. That is, there exists $a \in \mathbb{R}^m$ such that $a + U \subset V$. Inequality (2.6) implies that if, for a given $\theta \in \Theta$, there exists a convex, compact set S such that $S \subset_a \partial F_N(\theta)$ for all N, then $S \subset_a \partial F(\theta)$. Therefore, if such a set S exists with a positive probability then, by the interchangeability formulas (2.3) and (2.4), F is not differentiable at θ. This point is next presented in a rigorous way.

Condition 2.1: For a given $\theta_0 \in \Theta$, there exists a convex, compact set C, containing more than one point, and $p > 0$ such that

$$\mathbb{P}\{C \subset_a \partial g_m(\theta_0), \text{ for some } m \le \beta\} = p \qquad (2.7)$$

where $\beta = \beta(\theta_0)$ is the length of the first regenerative cycle of the set-valued process $\partial g_n(\theta_0)$.

This condition implies that, for some $m \le \beta = \beta(\theta_0)$, $g_m(\cdot)$ is nondifferentiable at θ_0 with probability at least $p > 0$. Condition 2.1 can be easily verifiable, and as we will show in the following proposition, it is sufficient for nondifferentiability of $F(\theta)$ at the point θ_0.

Proposition 2.2: Suppose that the functions $g_n(\theta)$ are convex, that for any $\theta \in \Theta$ the processes $g_n(\theta)$ and $\partial g_n(\theta)$ are regenerative with regenerative cycles having finite first-order moments and that Condition 2.1 holds. Then the mean steady-state function $F(\theta)$ is not differentiable at the point θ_0.

Proof: Condition 2.1 states that, with probability $p > 0$, $C \subset_a \partial g_m(\theta_0)$ for some $m \le \beta$. Since the subdifferential of a sum of convex functions is equal to the sum of the subdifferentials of these functions, it follows from Condition 2.1 that

$$\mathbb{P}\left\{C \subset_a \partial\left[\sum_{n=1}^{\beta} g_n(\theta_0)\right]\right\} \ge p.$$

Consider an integer $k > 0$ and denote by $N(k)$ the integer which terminates the kth regenerative cycle of the process $\partial g_n(\theta_0)$. Consider also an integer l greater than $\mathbb{E}\{\beta\}$ (recall that $\mathbb{E}\{\beta\}$ is assumed to be finite). By the Law of Large Numbers, $N(k)/k \to \mathbb{E}\{\beta\}$ in probability, and therefore, $\mathbb{P}\{N(k) < kl\} \to 1$ as $k \to \infty$. Set $N(0) := 0$ and for every $j = 1, 2, \cdots$, denote by A_j the event that, for some $m \in \{N(j-1)+1, \cdots, N(j)\}$, the inclusion $C \subset_a \partial g_m(\theta_0)$ holds. By Condition 2.1 and the regenerative property of the process $\partial g_n(\theta_0)$, we have that $\mathbb{P}\{A_j\} = p$ and the events A_j, $j = 1, 2, \cdots$, are mutually independent.

For any positive integer $r \le k$, let $B_{r,k}$ denote the event that A_j happens at least r times in the first k trials (regenerative cycles), and consider the probabilities $q_{r,k} := \mathbb{P}\{B_{r,k}\}$. Fix a positive number $q < p$ and take $r := [qk]$, i.e., r is the integer part of qk. We can assume (by increasing k if necessary) that $r > 0$. By the Central Limit Theorem for binomial distributions, and since $q < p$, we have that $q_{r,k} \to 1$ as $k \to \infty$.

To sum up the above arguments, we have the following. The event $B_{r,k}$ happens with probability $q_{r,k}$. The latter event means that for at least r different values of $j = 1, \cdots, k$, $C \subset_a \partial g_m(\theta_0)$ for some $m \in \{N(j-1)+1, \cdots, N(j)\}$. This implies that for at least r values of $j = 1, \cdots, k$, the inclusion $C \subset_a \partial[\sum_{n=N(j-1)+1}^{N(j)} g_n(\theta_0)]$ holds. Therefore, with probability at least $q_{r,k}$, we have $rC \subset_a \partial[\sum_{n=1}^{N(k)} g_n(\theta_0)]$.

Next, recall that $l > \mathbb{E}\{\beta\}$ and hence, $\mathbb{P}\{N(k) < kl\} \to 1$ as $k \to \infty$. Therefore, there exists k_0 such that, for every $k \ge k_0$

$$\mathbb{P}\left\{rC \subset_a \partial\left[\sum_{n=1}^{kl} g_n(\theta_0)\right]\right\} > q_{r,k}/2.$$

Since, $F_{kl}(\theta) = (kl)^{-1} \sum_{n=1}^{kl} \mathbb{E}\{g_n(\theta)\}$, it follows from the interchangeability formula (2.4) that

$$\tfrac{1}{2} q_{r,k}(r/kl) C \subset_a \partial F_{kl}(\theta_0).$$

Now, as $k \to \infty$, $q_{r,k} \to 1$, $r/k \to q$, and (by (2.5)), $F_{kl}(\theta) \to F(\theta)$. Therefore, and by formula (2.6) and the discussion that follows it, we get that

$$(2l)^{-1} qC \subset_a \partial F(\theta_0).$$

Consequently, the subdifferential $\partial F(\theta_0)$ contains more than one point, and hence, F is not differentiable at θ_0.

The next section applies the above result to examples of single queues where, nondifferentiabilities at dense sets are established.

III. Examples

Consider a first-in/first-out (FIFO) G/G/1 queue whose service and interarrival times depend on a parameter $\theta \in \Theta$, with the parameter space Θ being an open subset of $\mathbb{R}^m$. For a fixed $\theta \in \Theta$, let $s_n(\theta)$ be the nth service time, and let $\tau_n(\theta)$ be the time between arrivals of the $(n-1)$th and the nth customers, $n = 1, 2, \cdots$. We assume that the first customer arrives at an empty queue, that the functions $s_n(\theta)$ and $\tau_n(\theta)$ are continuously differentiable, that for a fixed θ the processes $\tau_n(\theta)$ and $s_n(\theta)$ are stationary and ergodic, and that for every $\theta \in \Theta$ the queue is regenerative with the expected number of customers served in one busy period (regenerative cycle) being finite.

Let $g_n(\theta)$ denote the nth sojourn time. We have then that the sojourn time process has a unique stationary and ergodic distribution and that the long-run average functions

$$f_N(\theta) := N^{-1} \sum_{n=1}^{N} g_n(\theta)$$

converge (pointwise) w.p.1 to the expected-value (mean) steady state sojourn time $F(\theta)$. A recursion relation between the sojourn times is given by the Lindley equation [7]

$$g_n(\theta) = s_n(\theta) + [g_{n-1}(\theta) - \tau_n(\theta)]_+. \qquad (3.1)$$

If the functions $s_n(\theta)$ and $\tau_n(\theta)$, $n = 1, 2, \cdots$, are convex and concave, respectively, then it follows by induction from (3.1) that the functions $g_n(\theta)$, and hence $f_N(\theta)$ and $F(\theta)$, are also convex (cf. [11], [12]).

We next discuss two examples where the parameter θ is a scalar, Θ is an open interval of the real line and the service times $s_n(\theta)$ are convex, differentiable functions of θ while the interarrival times τ_n do not depend on θ. Denote by $l_n(\theta)$ the integer l such that the lth customer starts the busy cycle to which the nth customer belongs. It follows then that if $dg_n(\theta)/d\theta$ exists then (see [13])

$$\frac{dg_n(\theta)}{d\theta} = \sum_{i=l_n(\theta)}^{n} \frac{ds_i(\theta)}{d\theta}. \qquad (3.2)$$

A similar formula holds for the one-sided derivatives. In the forthcoming examples, $ds_i(\theta)/d\theta \ge \xi(\theta)$, where $\xi(\theta)$ is a positive valued function of θ. Consequently, by (3.1), $g_n(\cdot)$ is not differentiable at those points θ where $g_{n-1}(\theta) = \tau_n$. Moreover, at those points the subdifferential $\partial g_n(\theta)$ contains, up to an additive constant, the interval $[0, \xi(\theta)]$. Denote by a_n and $d_n(\theta)$ the nth arrival and departure times, respectively. Clearly, the condition $\tau_n = g_{n-1}(\theta)$ is equivalent to $a_n = d_{n-1}(\theta)$. Therefore Condition 2.1 holds at a point θ_0 if, with positive probability, the event $a_n = d_{n-1}(\theta_0)$ happens in the first busy period. This, in turn, will imply nondifferentiability of the mean steady-state function $F(\theta)$ at θ_0.

Example 3.1: Consider a G/D/1 queue with $s_n(\theta) = \theta$. Suppose that the distributions of the interarrival times τ_n have atoms at two points b and c, with $b < c$, and let $\Theta = (b, c)$. We assume that the queue is stable, and it has regenerative cycles with finite first moments for every $\theta \in \Theta$.

Proposition 3.1: Suppose that any finite sequence of interarrival times $\tau_1, \cdots, \tau_n$, $n = 1, 2, \cdots$, with $\tau_i = b$ or $\tau_i = c$, $i = 1, \cdots, n$, can happen in any busy period with some positive probability. Then for every pair of positive integers k and m, the mean steady-state function $F(\cdot)$ is not differentiable at the point

$$\theta = \frac{(k-1)b + mc}{k - 1 + m}. \tag{3.3}$$

Proof: Fix k and m and let θ be given as in (3.3). By Proposition 2.2 and the above discussion, it suffices to show that

$$\mathsf{P}\{d_{n-1}(\theta) = a_n\} > 0 \tag{3.4}$$

where $n = k + m$. Let A denote the event that $\tau_i = b$, $i = 2, \cdots, k$, and $\tau_i = c$, $i = k+1, \cdots, k+m$. We have that $\mathsf{P}\{A\} > 0$. We show now that if $\omega \in A$, then $d_{n-1}(\theta) = a_n$. This will establish (3.4).

Suppose, without loss of generality, that $a_1 = 0$. By assumption, $a_2 = b, \cdots, a_k = (k-1)b$. After arrival of the first k customers, the interarrival times change to c, and hence, $a_{k+1} = (k-1)b + c, \cdots, a_{k+m} = (k-1)b + mc$. In particular, $a_n = (k-1)b + mc$ (since $n = k + m$). Now consider the departure times. The first customer arrives at an empty queue and hence, $d_1(\theta) = \theta$. The first k customers have increasing waiting times (because $\theta > b$) and hence, $d_i(\theta) = i\theta$, $i = 1, \cdots, k$. Next, $\tau_i = c$, $i = k+1, \cdots, k+m$, therefore, and since $\theta < c$, the waiting times of successive customers (starting from the $(k+1)$th customer) are monotone-decreasing until the queue becomes empty. By (3.3), for every $j = 1, \cdots, m-1$, $a_{k+j} = (k-1)b + jc < (k-1+j)\theta = d_{k-1+j}(\theta)$, and hence, $d_{k+j}(\theta) = (k+j)\theta$. In particular, for $j = m-1$, $d_{k-1+m}(\theta) = (k-1+m)\theta$. By (3.3), and the fact that $n = k + m$, we have that

$$d_{n-1}(\theta) = d_{k-1+m}(\theta) = (k-1)b + mc.$$

On the other hand, we have seen that $a_n = (k-1)b + mc$.

We thus have seen that, for every $\omega \in A$, $d_{n-1}(\theta) = a_n$, and since $\mathsf{P}\{A\} > 0$, formula (3.4) is established. This completes the proof.

Let us make the following remarks. The set of points θ in the form (3.3) taken over all possible integers $k > 0$ and $m > 0$, is dense in the interval $\Theta = (b, c)$. The assumption that $s_n(\theta) = \theta$ naturally can be relaxed; service times with atoms at θ often will suffice. Interarrival times with two atoms, as above, can often happen to a queue in a network whose inputs include two queues with different atoms in their service times.

The presence of two atoms in the interarrival times distribution was crucial for the occurrence of the nondifferentiabilities at a dense set. We now present an example where the interarrival times have only one atom, and yet, the mean delay is nondifferentiable at a dense set. What makes it happen here is a second input stream whose interarrivals are supported on the half-line $\mathbb{R}^+$. This example was discussed in [16], where the nondifferentiability of $F(\theta)$ was conjectured.

Example 3.2: Consider the network shown in Fig. 1. Both queues are FIFO and have infinite buffers, and the two input processes from S_1 and S_2 are Poisson and mutually independent. Let Q_1 have deterministic server with service times of b seconds, and therefore the interarrival times to Q_2 from Q_1 have an atom at b. The input process to Q_2 consists of the arrivals from Q_1 and the independent source S_2, and hence, the interarrival times to Q_2 have an atom at b. Suppose that the service times at Q_2 are deterministic and their value is θ, the parameter of variation. Let $\theta < b$. It was shown in [16] that, if θ/b is rational then Condition 2.1 is satisfied with regard to Q_2. In fact, it was shown that, starting with an empty network, there exists an integer n such that

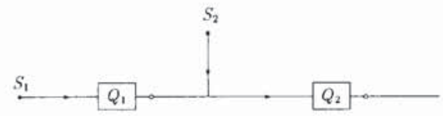

Fig. 1. Queueing network with two queues in tandem and two sources.

$$\mathsf{P}\{d_{n-1}(\theta) = a_n\} > 0 \tag{3.5}$$

$d_{n-1}(\theta)$ is the $(n-1)$th departure from Q_2 and a_n is the nth arrival to Q_2, and n depends on the ratio θ/b. As a result, the expected-value sojourn time at Q_2, denoted by $F(\theta)$, is not differentiable at every point θ such that θ/b is rational.

The details of the proof of formula (3.5) can be found in [16] but, to illustrate the main point, we present here a particular case. Let $\theta = (2/5)b$. We now define an event A having a positive probability such that, for every $\omega \in A$, $b_5(\theta) = a_6$. The event A is defined by the arrival schedule of the first six customers to Q_2 in such a way that, the first customer arrives at an empty queue, and the second-to-fifth customers arrive when the queue is nonempty. Consequently, for $i = 2, \cdots, 5$, $d_i(\theta) = d_1(\theta) + (i-1)\theta$. In particular, $d_5(\theta) = d_1(\theta) + 4\theta$, and it is at this time that the sixth customer arrives.

The event A is defined as follows. The first customer to Q_2 comes from Q_1 at time $\alpha + b$ (α is its arrival time to Q_1, and b is its service time there); two additional customers arrive to Q_2 from Q_1 at times $\alpha + 2b$ and $\alpha + 3b$, respectively. We denote these three customers by C_{1j}, $j = 1, 2, 3$, where the subscript means that it is the jth customer arriving to Q_2 from Q_1. Next, three customers arrive to Q_2 from S_2 at some times τ_j, $j = 1, 2, 3$, where $\tau_1 \in (\alpha + b, \alpha + b + \theta)$, $\tau_2 \in (\tau_1, \alpha + b + 2\theta)$, and $\tau_3 \in (\alpha + 2b, \alpha + 2b + \theta)$.

Now it is clear that $\mathsf{P}\{A\} > 0$ because of the facts that the interarrival times from S_2 have a density supported on $\mathbb{R}^+$ and the interarrival times from Q_1 have an atom at b. We next show that, if $\omega \in A$ then $d_5(\theta) = a_6$.

Fix $\omega \in A$. The first arrival to Q_2 is C_{11}, and its arrival time is $a_1 = \alpha + b$. The next two arrivals come from Q_2 at times $a_2 = \tau_1$ and $a_3 = \tau_2$, respectively. Since $\theta = 0.4b$, $\tau_1 \in (\alpha + b, \alpha + 1.4b)$ and $\tau_2 \in (\tau_1, \alpha + 1.8b)$. The fourth arrival is C_{12}, and its arrival time is $a_4 = \alpha + 2b$. The fifth arrival comes from S_2 at time $a_5 = \tau_3 \in (\alpha + 2b, \alpha + 2.4b)$. Finally, the sixth arrival is C_{13}, and it comes at time $a_6 = \alpha + 3b$. Next, since the service time at Q_2 is θ, the second-to-fifth customers arrive at a nonempty queue. Consequently, the departure times are given by $d_1(\theta) = a_1 + \theta$, and $d_i(\theta) = d_{i-1}(\theta) + \theta$, $i = 2, \cdots, 5$. In particular, $d_5(\theta) = d_1(\theta) + 4\theta = a_1 + 5\theta$. Since $\theta = 0.4b$ and $a_1 = \alpha + b$, we have that $d_5(\theta) = a_1 + 2b = \alpha + 3b$. But, we have seen that $a_6 = \alpha + 3b$, and hence, $b_5(\theta) = a_6$.

We remark that, as in the previous example, the assumptions made can be greatly relaxed. Generally, the arrival processes from S_1 and

S_2 only need densities supported on $\mathbb{R}^+$, the service times at Q_1 have to have an atom at a point b, and the service times at Q_2 have to have an atom at a point $c(\theta)$ such that $dc(\theta)/d\theta > 0$ for every θ.

IV. Conclusions

This paper established the nondifferentiability of certain steady-state functions in queueing networks where the sample performance functions are convex or, more generally, subdifferentiable. The analysis was carried out in three stages, yielding the following results: i) nondifferentiabilities of the sample performance functions can occur at dense sets in the parameter space, ii) these nondifferentiabilities are retained in the expected-value functions over finite horizons (by the interchangeability formula (2.4)), and iii) they are carried over to the steady state (via formula (2.5)).

Two case-study examples were discussed: the first is of a single queue whose interarrival times have two atoms, and the second consists of a two-queue serial network with two sources. These examples are quite simple and generic in the sense that they often comprise subsets of larger networks. Yet, they clearly exhibit the co-occurrence of two events in the sample path with positive probability at dense sets of points, which constitutes the crucial condition for nondifferentiability.

References

[1] S. Asmussen, *Applied Probability and Queues*. New York: Wiley, 1987.

[2] X. R. Cao, W. B. Gong, and Y. Wardi, "Ill-conditioned performance functions of queueing systems," in *Proc. 31st Conf. Dec. Contr.*, Tucson, AZ, 1992.

[3] P. Glasserman, *Gradient Estimation Via Perturbation Analysis*. Boston, MA: Kluwer, 1991.

[4] Y. C. Ho and X. R. Cao, *Perturbation Analysis of Discrete Event Dynamic Systems*. Boston, MA: Kluwer, 1991.

[5] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*. Amsterdam: North-Holland, 1979.

[6] J. Q. Hu, "Convexity of sample path performance and strong consistency of infinitesimal perturbation analysis estimates," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 258–262, 1992.

[7] D. V. Lindley, "The theory of queues with single server," in *Proc. Camb. Phil. Soc.*, vol. 48, 1952, pp. 277–289.

[8] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.

[9] R. T. Rockafellar and R. J. B. Wets, "On the interchange of subdifferentiation and conditional expectation for convex functionals," *Stochastics*, vol. 7, pp. 173–182, 1982.

[10] R. Y. Rubinstein and A. Shapiro, *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*. New York: Wiley, 1993.

[11] M. Shaked and J. G. Shantikumar, "Stochastic convexity and its applications," *Adv. Appl. Prob.*, vol. 20, pp. 427–466, 1988.

[12] J. G. Shantikumar and D. D. Yao, "Second-order stochastic properties in queueing systems," in *Proc. IEEE*, vol. 77, pp. 162–170, 1989.

[13] R. Suri, "Perturbation analysis: The state of the art and research issues explained via the GI/G/1 queue," in *Proc. IEEE*, vol. 77, pp. 114–137, 1989.

[14] R. Suri and B. R. Fu, "Using continuous flow models to enable rapid analysis and optimization of discrete production lines—A progress report," in *Proc. 19th Annual NSF Grantees Conf. Design Manufact. Syst. Res.*, Charlotte, NC, 1993.

[15] Y. Wardi and J. Q. Hu, "Strong consistency of infinitesimal perturbation analysis for tandem queueing networks," *J. Discrete Event Dynamic Syst.: Theory and Appl.*, vol. 1, pp. 37–59, 1991.

[16] Y. Wardi, M. W. McKinnon, and R. Schuckle, "On perturbation analysis of queueing networks with finitely supported service time distributions," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 863–867, 1991.

[17] R. W. Wolff, *Stochastic Modeling and the Theory of Queues*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

A Reduced-Order Model about Structural Wave Control Based upon the Concept of Degree of Controllability

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Abstract—**After introducing the concept and criteria of controllability and degree of controllability about structural wave control in this paper, we put forward a new approach of structural reduced order, which is similar to the constrained substructural method in dynamics, and is also the extension of the method of aggregation raised by Aoki in the 1960s [1]. Furthermore, this approach has the characteristic of clear in the meaning of physics and is easy to be realized.**

I. Introduction

The study on the controllability of system has matured since Kalman first put forward the concept in 1960. In the field of structural control, sets of strict theories and criteria have been established to determine the controllability of structure being expressed by either lumped or distributed parameters system. At present, more and more attention [2]–[6] are put on the study of wave control with the trend of space structure designed to larger size arising more flexibility.

As the basic characteristics of structure response, wave and vibration are quite different in qualities from the point of view of dynamics, it is based on this consideration that we define in this paper the controllability of structural wave domain control and the degree of controllability measuring it. Then the criterion of controllability is obtained concerning the above two definitions.

It is well known that the problem of reducing the order of the structural model is very important to the design of a control system, that is to say, a systematic method for approximate analysis and synthesis of controls of large-scale system should be present and sufficient information of the original system can be saved, so that the analysis and simulation of the large-scale system are derived with the simple model.

There are about two sorts of the conventional approaches about the reduced-order system.

1) Open-loop reduced order, which means the order of the mathematical model of the original system is directly reduced, and the usual design of system is obtained on the basis of the reduced-order model.
2) Closed-loop reduced order, that is, the order of the controller which is derived on the basis of the original system is directly reduced.

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