Graph Theory Nice Notes

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Contents

1 Basic Definitions

Definition 1.1 (Graph)**.** A **graph** G is composed of a **vertex** set $V(G)$ and an **edge** set $E(G) \subseteq [V(G)]^2$.

Definition 1.2 (Simple)**.** A graph G is **simple** if $E(G) \subseteq \binom{V(G)}{2}$; the graph has no loops or parallel edges.

Definition 1.3 (Connected)**.** A graph G is **connected** if there exists a path from u to v for all $u, v \in V(G)$.

Definition 1.4 (Degree)**.** The **degree** $d(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v.

Theorem 1 (Handshake Lemma)**.** The sum of the degrees of the vertices of a graph G is

$$\sum_{v \in V(G)} d(v) = 2 \cdot |E(G)|$$

Definition 1.5 (Graph Isomorphism)**.** A **graph isomorphism** from graph G to graph F is a one-to-one function $f : V(G) \to V(F)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(F)$ for all $u, v \in V(G)$.

Definition 1.6 (Graph Homomorphism)**.** A **graph homomorphism** from graph G to graph F is a function $f : V(G) \to V(F)$ such that $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(F)$.

1.1 Graph Properties

Definition 1.7 (Clique Number)**.** The **clique number** $\omega(G)$ of a graph G is the size of the largest fully connected component.

Definition 1.8 (Girth)**.** The **girth** $g(G)$ of a graph G is the length of the longest cycle.

Definition 1.9 ($\Delta(G)$, $\delta(G)$)**.** The largest and smallest degree of a vertex in a graph G are denoted

$$\Delta(G) = \max\{d(v) : v \in V\} \text{ and } \delta(G) = \min\{d(v) : v \in V\},$$

respectively.

1.2 Types of Graphs

Definition 1.10 (Complement)**.** The **complement** $\overline{G}$ of a simple graph G has

$$V(\overline{G}) = V(G) \qquad E(\overline{G}) = \binom{V(G)}{2} \setminus E(G).$$

Definition 1.11 (Line Graph)**.** The **line graph** $L(G)$ of a graph G has

$$V(L(G)) = E(G) \qquad E(L(G)) = \{\{e, f\} : e, f \in E(G) \text{ and } e \cap f \neq \emptyset\}.$$

Definition 1.12 (Path)**.** A **path graph** P_n has

$$V(P_n) = \{0, 1, \ldots, n\} \qquad E(P_n) = \{\{i, i+1\} : 0 \leq i < n\}.$$

Definition 1.13 (Cycle)**.** A **cycle graph** C_n has

$$V(C_n) = \{0, 1, \ldots, n\} \qquad E(C_n) = \{\{i, j\} : i + 1 \equiv j \pmod{n}\}.$$

Definition 1.14 (Kneser Graph)**.** A Kneser graph $\mathrm{KG}(n, k)$ has

$$V(\mathrm{KG}(n, k)) = \binom{[n]}{k} \qquad E(\mathrm{KG}(n, k)) = \{\{A, B\} : A \cap B = \emptyset\}$$

1.3 Trees

Definition 1.15 (Tree)**.** A **tree** is a connected graph that contains no cycles.

Theorem 2. Every tree on at least two vertices contains at least two vertices with degree 1.

Theorem 3. A tree on n vertices always has $n-1$ edges.

Theorem 4 (Cayley)**.** The number of different labeled trees on n vertices is n^{n-2}.

1.4 Eulerarian Cycles

Definition 1.16 (Eularian Circuit)**.** A closed walk containing every edge of a graph G exactly once is a called a **closed Eularian tour** or **Eularian circuit**.

Definition 1.17. A walk containing every edge exactly one but ending at a different vertex than where it started is called an **open Eularian tour** or **Eularian path**.

Theorem 5 (Euler)**.** A graph contains a closed Eularian tour if and only if it is connected and all its degrees are even.

Corollary 1. A graph G contains an open Eularian tour if and only if it is connected and exactly two of its degrees are odd.

2 Hamiltonian Cycles

Definition 2.1 (Hamiltonian Cycle)**.** A cycle [resp. path] in a graph which contains every vertex exactly once is called a **Hamiltonian cycle** [resp. **path**].

Remark 1. If a graph G has a Hamiltonian cycle, then for $k \in \mathbb{N}$ if we delete k vertices of G then G cannot fall apart into more than k components.

Theorem 6 (Dirac)**.** If $d(v) \geq \frac{|V(G)|}{2}$ for all $v \in V(G)$, then G has a Hamiltonian cycle.

Theorem 7 (Ore)**.** If $d(x) + d(y) \geq |V(G)|$ for all $x, y \in V(G)$ such that $\{x, y\} \notin E(G)$, then G has a Hamiltonian cycle.

Theorem 8 (Pósa)**.** Let $G = (V, E)$ be a graph with $|V| = n$. Order the vertices of G such that

$$d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n).$$

If $d(v_k) \geq k + 1$ for every $k < \frac{n}{2}$, then G contains a Hamiltonian cycle.

Theorem 9 (Chvátal)**.** Let G be a graph with $|V(G)| = n$. Order the vertices of G such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$.

1. If $d(v_k) \leq k < \frac{n}{2}$ implies $d(v_{n-k}) \geq n - k$, then G has a Hamiltonian cycle.
2. Otherwise, there exists an n-vertex graph G' with degrees $d(v_1') \leq d(v_2') \leq \cdots \leq d(v_n')$ for which $d(v_i') \geq d(v_i)$ for every i and G' has no Hamiltonian cycle.

3 Matching

Definition 3.1 (Matching)**.** A **matching** is a set of independent (pairwise non-incident) edges in a graph.

Definition 3.2 (Perfect Matching)**.** A **perfect matching** is matching covering all edges.

Theorem 10 (Tutte)**.** Let $c_{\text{odd}}(F)$ denote the number of components of F with an odd number of vertices. A graph G has a perfect matching if and only if $c_{\text{odd}}(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

3.1 Gallai's Identity

Definition 3.3 (Matching Number)**.** The **matching number** $\nu(G)$ of a graph G is the size of the largest matching in G.

Definition 3.4 (Stable Set Number)**.** The **stable set number** $\alpha(G)$ of a graph G is the largest independent set of vertices in G.

Definition 3.5 (Vertex Cover Number)**.** The **vertex covering number** $\tau(G)$ of a graph G is the minimum number of vertices that cover all edges.

Definition 3.6 (Edge Cover Number)**.** The **edge cover number** $\rho(G)$ of a graph G is the minimum number of edges that cover all vertices.

Remark 2. For all G, $\tau(G) \geq \nu(G)$ and $\rho(G) \geq \alpha(G)$.

Theorem 11 (Gallai)**.** If G is a graph without isolated vertices, then

$$\alpha(G) + \tau(G) = |V(G)| = \nu(G) + \rho(G).$$

3.2 Bipartite Graphs

Definition 3.7 (Bipartite)**.** A graph $G = (A, B, E)$ is called **bipartite** if $V(G) = A \cup B$ and $E(G) \subseteq A \times B$.

Theorem 12. A graph is bipartite if and only if it has no subgraph isomorphic to C_{2k+1} for any $k \geq 1$.

Theorem 13 (Hall)**.** If $G = (A, B, E)$ is a bipartite graph, then there exists a matching in G covering A if and only if $|N(U)| \geq |U|$ for all $U \subseteq A$.

Theorem 14 (Frobenius)**.** A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if $|N(U)| \geq |U|$ for all $U \subseteq A$ and $|A| = |B|$.

Theorem 15 (König)**.** If G is bipartite, then $\tau(G) = \nu(G)$. Additionally, $\alpha(G) = \rho(G)$ if G has no isolated vertices.

3.3 Stable Matching

Definition 3.8 (Stable Matching)**.** Given a graph G with a preference list of the neighbors of every vertex v, a **stable matching** is a matching M such that there does not exists an edge $\{u, v\} \in E(G) \setminus M$ where both u and v compare each compared to their partner in M.

Theorem 16 (Gale-Shapley)**.** If G is bipartite, then there always exists a stable matching.

4 Coloring

4.1 Vertex Coloring

Definition 4.1 (Vertex Coloring)**.** A **vertex coloring** of the graph G is a function

$$c : V(G) \to S$$

such that $c(v) = c(w)$ if and only if $\{c(v), c(w)\} \notin E(G)$ for all $v, w \in V(G)$.

Definition 4.2. The **chromatic number** $\chi(G)$ is the minimum number of colors needed for a proper coloring of the vertices of G:

$$\chi(G) = \min\{|S| : \text{there exists a vertex coloring } c : V(G) \to S\}.$$

Remark 3. A vertex coloring with n colors of a graph G is a homomorphism from G to K_n. Then

$$\chi(G) = \min\{n : F \to K_n\}.$$

Example 4.1. The chromatic number of the Kneser graph is $\chi(\mathrm{KG}(n, k)) = n - 2k + 1$.

4.1.1 Bounds

Theorem 17. For a graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem 18 (Brooks)**.** If G is connected and $G \ncong K_n$ and $G \ncong C_{2k+1}$, then $\chi(G) \leq \Delta(G)$.

Theorem 19. For a graph G, $\chi(G) \leq \max\{\delta(G') : G' \subseteq G\} + 1$.

Theorem 20. For all $n \in \mathbb{N}$, there exists a graph G such that $\omega(G) = 2$ and $\chi(G) \geq n$.

Theorem 21 (Erdős)**.** For all $k, l \in \mathbb{N}$, there exists a graph G such that $g(G) \geq l$ and $\chi(G) \geq k$.

4.2 Edge Coloring

Definition 4.3 (Edge Coloring)**.** An **edge coloring** of the graph G is a function

$$c : E(G) \to S$$

such that $c(e) \neq c(f)$ for any adjacent edges $e, f \in E(G)$.

Definition 4.4. The **edge-chromatic number** $\chi_e(G)$ is the minimum number of colors needed for a proper coloring of the edges of G:

$$\chi_e(G) = \min\{|S| : \text{there exists an edge coloring } c : V(G) \to S\}.$$

Remark 4. Since adjacent edges must be colored distinct colors, $\chi_e(G) \geq \Delta(G)$ for all graphs G.

Theorem 22 (Shannon)**.** If G is a finite graph, then $\chi_e(G) \leq \frac{3}{2}\Delta(G)$.

Theorem 23 (Vizing)**.** If G is a finite simple graph, then $\chi_e(G) \leq \Delta(G) + 1$.

4.3 List Coloring

Definition 4.5 (List Coloring)**.** Given a graph G and sets of colors $L(v)$ for all $v \in V(G)$, a **list coloring** is a vertex coloring

$$c : V(G) \to \bigcup_{v \in V(G)} L(v)$$

such that $c(v) \in L(v)$ for all $v \in V(G)$.

Definition 4.6 (Choice Number)**.** The **choice number** $\text{ch}(G)$ is the minimum $k \in \mathbb{N}$ such that if $L(v) \geq k$ for all $v \in V(G)$, then there exists a list coloring $c : V(G) \to \bigcup_{v \in V(G)} L(v)$.

4.3.1 Bounds

Remark 5. For a graph G, $\text{ch}(G) \leq \max\{\delta(G') : G' \subseteq G\}$ and $\text{ch}(G) \leq \Delta(G) + 1$.

Theorem 24. For a graph G, $\text{ch}(G) \geq \chi(G)$.

Conjecture 1. If G is the line graph of some graph, then $\text{ch}(G) = \chi(G)$.

Theorem 25 (Galvin)**.** If G is the line graph *of a bipartite graph*, then $\text{ch}(G) = \chi(G)$.

Theorem 26. For every k, there exists a graph G such that $\chi(G) = 2$ and $\text{ch}(G) > k$.

4.4 Perfect Graphs

Definition 4.7. A graph is **perfect** if $\chi(G') = \omega(G')$ for all induced subgraphs G' of G.

Theorem 27 (Perfect Graph Theorem)**.** A graph is perfect if and only if its complement is perfect.

Theorem 28 (Strong Perfect Graph Theorem)**.** A graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.

Theorem 29. A graph G is perfect if and only if $\alpha(G') \cdot \omega(G') \geq |V(G')|$ for all induced subgraphs G' of G.

Theorem 30 (Replication Lemma)**.** Duplicating a vertex in a perfect graph and connecting the original vertex to its duplicated copy, we obtain a perfect graph.

4.5 Fractional Coloring

Definition 4.8 (Fractional Coloring)**.** Let $S(G)$ be the set of independent sets of a graph G. A **fractional coloring** of G is a function $f : S(G) \to \mathbb{R}_{\geq 0}$ such that

$$\sum_{A \ni v, A \in S(G)} f(a) \geq 1$$

for all $v \in V(G)$.

Definition 4.9 (Fractional Chromatic Number)**.** The **fractional chromatic number** $\chi_f(G)$ is the value

$$\min \sum_{A \in S(G)} f(A).$$

Definition 4.10 (Fractional Clique)**.** A **fractional clique** of a graph G is a nonnegative real function on the vertices of G such that sum of the values on the vertices of any independent set is at most one.

Definition 4.11 (Fractional Clique Number)**.** The **fractional clique number** $\omega_f(G)$ of a graph G is the maximum possible weight of a fractional clique. The **weight** of a fractional clique is the sum of the values on all vertices.

Theorem 31. For all graphs G, $\chi(f) \geq \chi_f(G) = \omega_f(G) \geq \omega(G)$.

Theorem 32. For all graphs G, $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Definition 4.12 (Vertex-Transitive)**.** A graph G is called **vertex-transitive** if there exists an automorphism $f : G \to G$ such that $f(u) = v$ for all $u, v \in V(G)$.

Theorem 33. If a graph G is vertex-transitive, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$.

Example 4.2. The fractional coloring number of the Kneser graph is $\chi_f(\mathrm{KG}(n,k)) = \frac{n}{k}$.

4.5.1 b-Fold Coloring

Definition 4.13 (b-Fold Coloring)**.** A **b-fold coloring** of a graph G with m colors is a function $f : V(G) \to \binom{[m]}{b}$ such that if $u, v \in V(G)$ are adjacent, then $f(u) \cap f(v) = \emptyset$.

Definition 4.14 (b-Fold Chromatic Number)**.** The **b-fold chromatic number** $\chi_b(G)$ is the minimum m such that a b-fold coloring with m colors is possible.

Remark 6. In an optimal b-fold coloring, if we weight every color class with weight $\frac{1}{b}$, we get a fractional coloring with total weight $\frac{\chi_b(G)}{b}$. Therefore, $\chi_f(G) \leq \frac{\chi_b(G)}{b}$.

Theorem 34. For a graph G,

$$\chi_f(G) = \inf \left\{ \frac{\chi_b(G)}{b} : b \in \mathbb{N} \right\}.$$

5 Planar Graphs

Definition 5.1 (Planar)**.** A graph is called **planar** if it can be drawn on the plane with no edge crossings.

Theorem 35 (Euler)**.** If G is a connected planar graph with with n vertices, e edges, and f faces, then

$$n + f = e + 2.$$

Remark 7. The vertices and edges of a convex polyhedron always form a planar graph.

Theorem 36. If G is a planar graph with n vertices and e edges, then

$$e \leq 3n - 6.$$

Corollary 2 (K_5)**.** The complete graph K_5 is **not** planar.

Theorem 37. If G is a triangle-free planar graph with n vertices and e edges, then

$$e \leq 2n - 4.$$

Corollary 3 ($K_{3,3}$)**.** The complete bipartite graph $K_{3,3}$ is **not** planar.

5.1 Forbidden Subgraphs

Theorem 38 (Kuratowski)**.** A graph is planar if and only if it does not contain a subgraph topologically isomorphic to K_5 or $K_{3,3}$

Definition 5.2. A graph F is a **minor** of graph G if F can be obtained from G be performing a sequence of the following operations:

- deleting vertices,
- deleting edges, and
- contracting edges.

Theorem 39 (Wagner)**.** A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.

5.2 Coloring Planar Graphs

Theorem 40 (Four Color Theorem)**.** If G is planar, then $\chi(G) \leq 4$.

Theorem 41 (Thomassen)**.** If a graph G is planar, then $\mathrm{ch}(G) \leq 5$

6 Extremal Graph Theory

Definition 6.1 (Extremal)**.** A graph $G \not\supseteq H$ on n vertices with the largest possible number of edges is called **extremal** for n and H. The number of edges of G is denoted $\mathrm{ex}(n, H)$:

$$\mathrm{ex}(n, H) = \max\{|E(G)| : H \not\subseteq G \text{ and } |V(G)| = n\}.$$

Definition 6.2 (Turán Graph)**.** The **Turan graph** $T_n(r-1)$ is an $(r-1)$-partite graph on n vertices whose partition sets differ in size by at most 1.

Theorem 42 (Turán)**.** For all integers r, n with $n \geq r > 1$, every graph $G \not\supseteq K^r$ with n vertices and $\mathrm{ex}(n, K^r)$ edges is the Turan graph $T_n(r-1)$.

Theorem 43 (Erdős-Stone)**.** For all integers $r \geq 2$ and $s \geq 1$, and every $\epsilon > 0$, there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least $|E(T_n(r-1))| + \epsilon n^2$ edges contains K_s^r as a subgraph.

Theorem 44 (Erdős-Simonovits)**.** For every graph G with at least one edge,

$$\lim_{n \to \infty} \mathrm{ex}(n, G) \binom{n}{2}^{-1} = \frac{\chi(G) - 2}{\chi(G) - 1}.$$

Theorem 45 (Bipartite Graphs)**.** For some constant c and $r = \min\{r, s\}$,

$$\mathrm{ex}(n, K_{r,s}) \leq c \cdot n^{2 - \frac{1}{r}}.$$

Theorem 46 (Even Cycles)**.** For some constant c,

$$\mathrm{ex}(n, C_{2k}) \leq c \cdot n^{1+\frac{1}{k}}$$

7 Ramsey Theory

Definition 7.1 (Ramsey Number)**.** The **Ramsey number** $R(k)$ is the smallest $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then there is a monochromatic K_k in any 2-edge-coloring of K_n.

Theorem 47 (Ramsey)**.** For every $k \in \mathbb{N}$, the Ramsey number $R(k)$ exists.

Definition 7.2 ($R(k_1, \ldots, k_r)$)**.** The **Ramsey number** $R(k_1, \ldots, k_r)$ is the smallest $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then for any r-edge-coloring of K_n there is a monochromatic K_{k_i} in the ith color.

Remark 8. It has been proven that $R(3) = 6$ and $R(4) = 18$. Larger Ramsey numbers are bounded, but their values are not known. One such bound is $\sqrt{2}^k < R(k) < 4^k$.

Theorem 48 (Nešetřil-Rödl)**.** For any graph H there exists a graph G such that in any 2-edge-coloring of G a monochromatic induced copy of H must occur. Furthermore, there exists a G with this property such that $\omega(G) = \omega(H)$.

8 Shannon Capacity

Definition 8.1 (OR-Product)**.** The **OR-product** $F \cdot G$ of two graph F and G is defined by

$$V(F \cdot G) = V(F) \times V(G),$$
$$E(F \cdot G) = \{(f, g), (f', g')\} : \{f, f'\} \in E(F) \text{ or } \{g, g'\} \in E(G)\}.$$

The t**-fold OR-product** of G with itself is denoted G^t.

Definition 8.2 (Shannon OR-Capacity)**.** The **Shannon OR-capacity** of a graph G is

$$C_{\mathrm{OR}}(G) = \lim_{t \to \infty} \sqrt[t]{\omega(G^t)}.$$

Remark 9. For a graph G, $\omega(G^t) \geq [\omega(G)]^t$ and $C_{\mathrm{OR}}(G) \geq \omega(G)$.

8.1 Bounds

Theorem 49. Let $p(G)$ be a graph parameter having the following properties:

1. $p(G) \geq \omega(G)$ for all G,
2. $p(F \cdot G) \leq p(F) \cdot p(G)$ for all F, G.

Then $C_{\mathrm{OR}} \leq p(G)$.

Corollary 4. For a graph G, $C_{\mathrm{OR}}(G) \leq \chi(G)$ and $C_{\mathrm{OR}}(G) \leq \chi_f(G)$.

8.2 Lovász Numbers

Definition 8.3 (Orthonormal Corepresentation)**.** An **orthonormal corepresentation** of a graph G is an attachment of unit vectors (in $\mathbb{R}^d$ for some d) to the vertices of G such that if u_i, u_j are the vectors attached to vertices $i, j \in V(G)$ and $\{i, j\} \in E(G)$, then u_i is orthogonal to u_j, $u_i \cdot u_j = 0$.

Definition 8.4. The **Lovász $\overline{\vartheta}$-number** of G is defined as

$$\overline{\vartheta}(G) = \min_{\substack{(u_1, \ldots, u_n) \\ \text{is an orthonormal} \\ \text{corep. of } G}} \min_{\substack{c \in \mathbb{R}^d \\ |c|=1}} \max_{i \in V(G)} \frac{1}{(c \cdot u_i)^2}.$$

Theorem 50. For a graph G, $C_{\mathrm{OR}} \leq \overline{\vartheta}(G)$.