EE-2200 Fall-98

Lecture 15

IIR Filters: Feedback

20-Nov-98

Info: Web-CT, Lab, HW

- Calendar:
  - Final Exam is Period 15 (ugh!)
  - Quiz Solutions are posted
- Grade Weightings will be posted
- Prob Set #7 will be posted Monday
  - Prob–Set #7 due on last day
- Lab #9: IIR Filters

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 249–263
- Other Reading:
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, pp. 269–282

NT PROBLEM

- iNTj
- eSFp

LECTURE #14 (Previous)

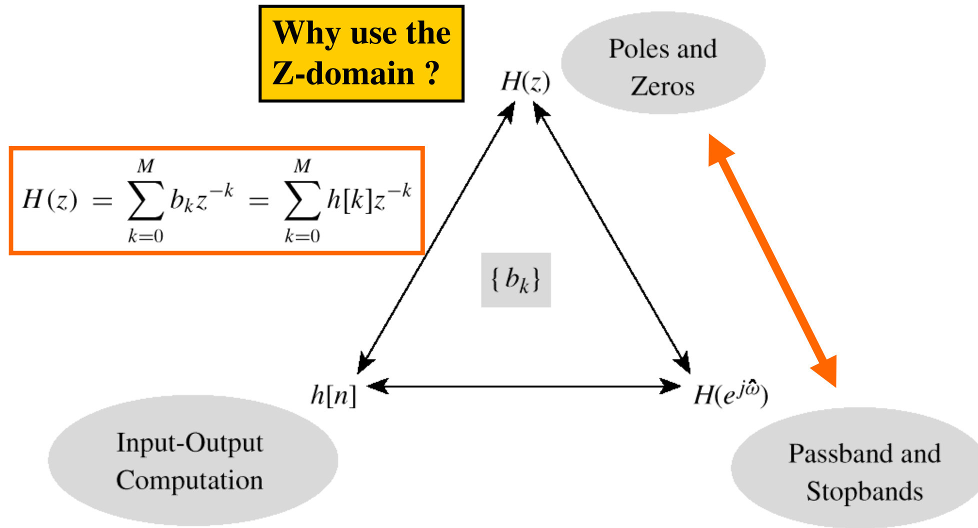
- Relate H(z) to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE DOMAINS:
  - Show Relationship for FIR:

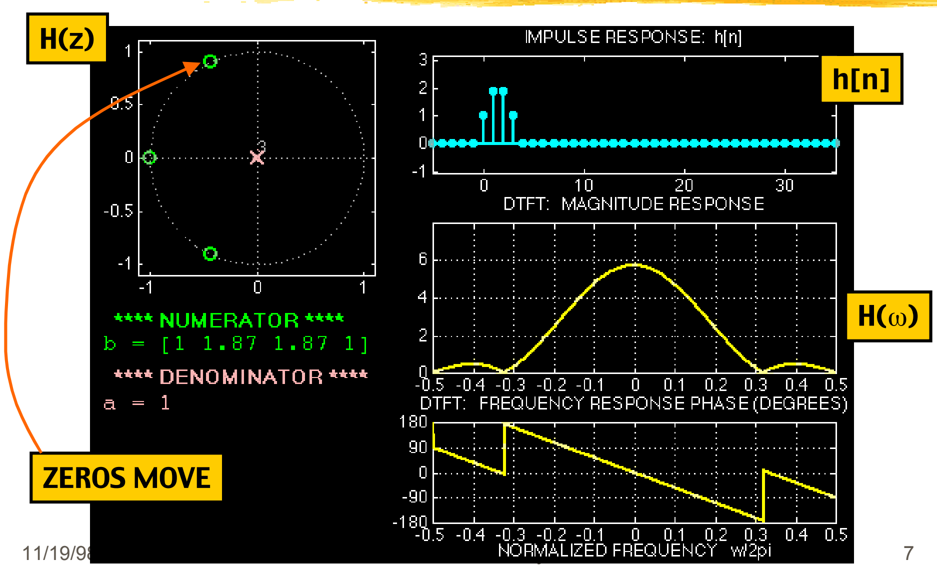
$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{b_k\}$ play a central role.

3 DOMAINS MOVIE: FIR



LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define IIR Filters
  - Have FEEDBACK: PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

  - Show how to compute the output y[n]
    - FIRST-ORDER CASE (N=1)
    - h[n] <--> H(z)

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

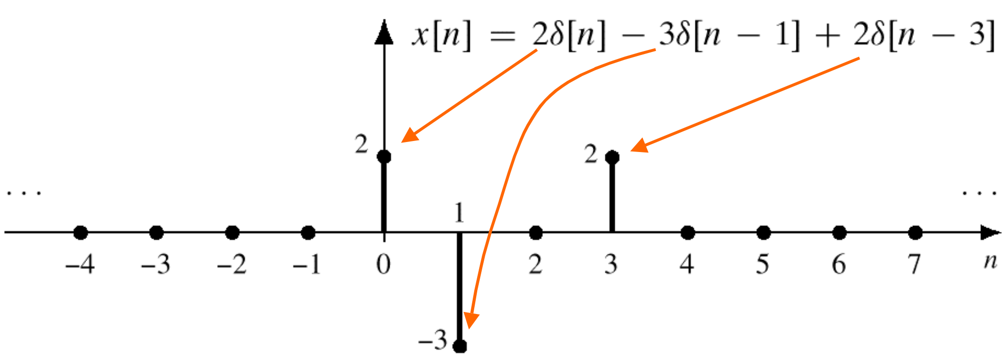
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$



COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED y[−1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- APPLIES TO ALL FEEDBACK TERMS

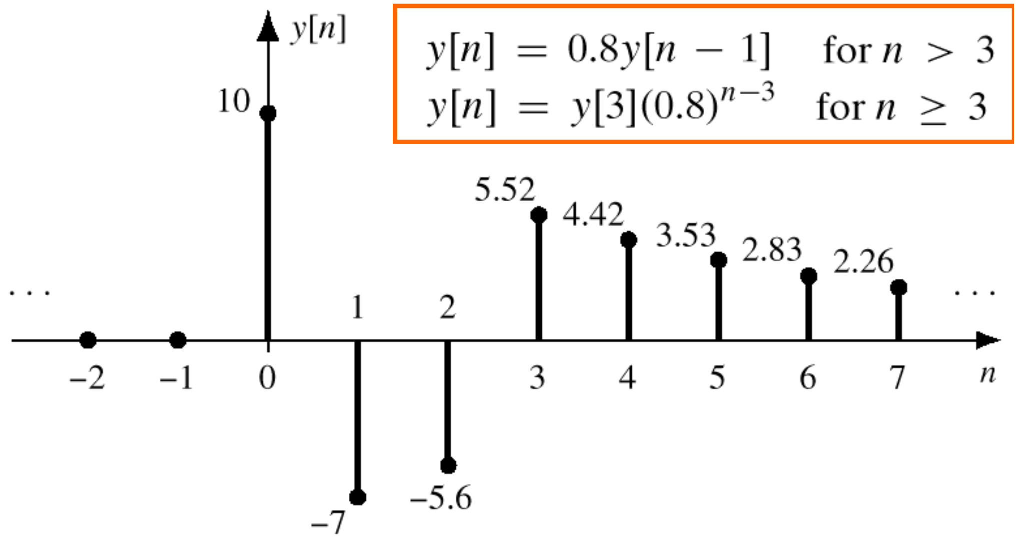
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$
$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$
$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]



$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$

IMPULSE RESPONSE

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad \boxed{h[n] = b_0(a_1)^n u[n]}$$

$$\boxed{u[n] = 1, \quad \text{for } n \geq 0}$$

STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$\boxed{u[n] = 1, \quad \text{for } n \geq 0}$$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
.		.

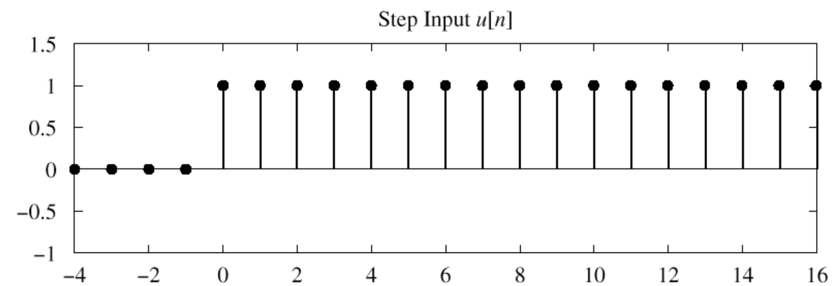
DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

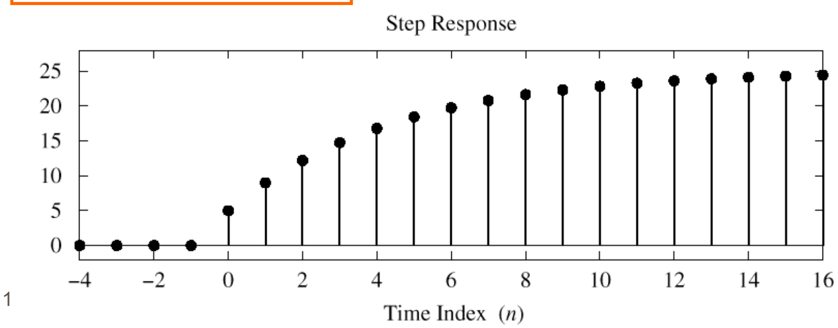
$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE



$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$



SUPERPOSITION

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

Because this system is linear and time-invariant, it follows that its impulse response can be thought of as a sum of two terms as in

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

$$= \begin{cases} 0 & n < 0 \\ b_0 & n = 0 \\ (b_0 + b_1 a_1^{-1})(a_1)^n & n \geq 1 \end{cases}$$

SUPERPOSITION

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$y[n] = y_1[n] + y_2[n]$$

where

$$y_1[n] = a_1 y_1[n-1] + b_0 x[n] \longrightarrow b_0 h[n]$$

$$y_2[n] = a_1 y_2[n-1] + b_1 x[n-1] \longrightarrow b_1 h[n-1]$$

Z-Transform of IIR Filter

- **DERIVE the SYSTEM FUNCTION H(z)**
  - **Use DELAY PROPERTY**

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- **NOTE the FILTER COEFFICIENTS**

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}} = \frac{B(z)}{A(z)}$$

POP QUIZ

- **Given:** $H(\hat{\omega}) = e^{-j\hat{\omega}} \cos(\hat{\omega})$
- **Find the Impulse Response, h[n]**
- **Find the output, y[n]**
  - **When** $x[n] = \cos(0.25\pi n)$

POP QUIZ: Answer #1

- **Find the Impulse Response, h[n]**

$$H(\hat{\omega}) = e^{-j\hat{\omega}} \cos(\hat{\omega})$$

$$= e^{-j\hat{\omega}}\left(\tfrac{1}{2}e^{j\hat{\omega}} + \tfrac{1}{2}e^{-j\hat{\omega}}\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2}e^{-j2\hat{\omega}}$$

$$\Rightarrow h[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-2]$$

POP QUIZ : Answer #2

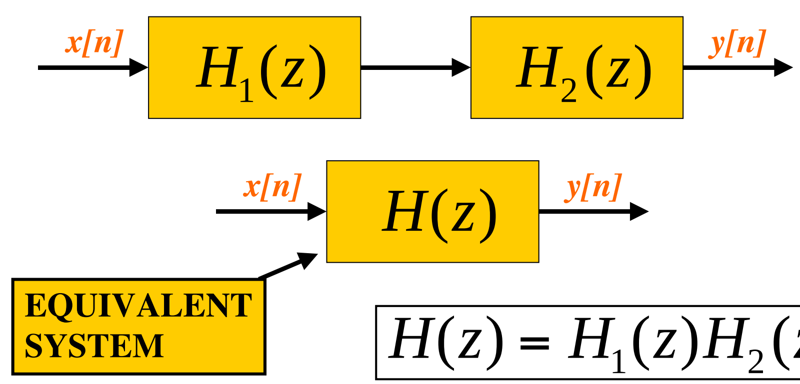
- **Find y[n] when** $\boxed{x[n] = \cos(0.25\pi n)}$

$$\boxed{\begin{aligned} y[n] &= |H|\cos(0.25\pi n + \angle H) \\ &= 0.707\cos(0.25\pi n - \tfrac{\pi}{4}) \end{aligned}}$$

$$\boxed{\begin{aligned} &H(\hat{\omega}) = e^{-j\hat{\omega}} \cos(\hat{\omega}) \qquad \boxed{\text{at } \hat{\omega} = \tfrac{\pi}{4}} \\ &H(\tfrac{\pi}{4}) = e^{-j\pi/4}\cos(\tfrac{\pi}{4}) = 0.707e^{-j\pi/4} \end{aligned}}$$

CASCADE EQUIVALENT

- **Multiply the System Functions**



$$\boxed{H(z) = H_1(z)H_2(z)}$$