EE-2200 Fall-98

Lecture 17

2nd-ORDER SYSTEMS

30-Nov-98

Info: Web-CT, Lab, HW

- Calendar:
  - Final Exam is Period 15
  - Review Session(s) planned
- Grade Weightings are posted
- Prob Set #7 is due Friday
- Lab #9 (this week) is short
  - FORMAL Lab Report Template posted

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 279–300
- Other Reading:
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, all

LECTURE OBJECTIVES

- SECOND–ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS
- THREE–DOMAIN APPROACH
  - UNIFIES h[n] & FREQUENCY RESPONSE in terms of POLES and ZEROS

THREE DOMAINS

$$H(z)=\frac{\sum b_k z^{-k}}{1+\sum a_\ell z^{-\ell}}$$

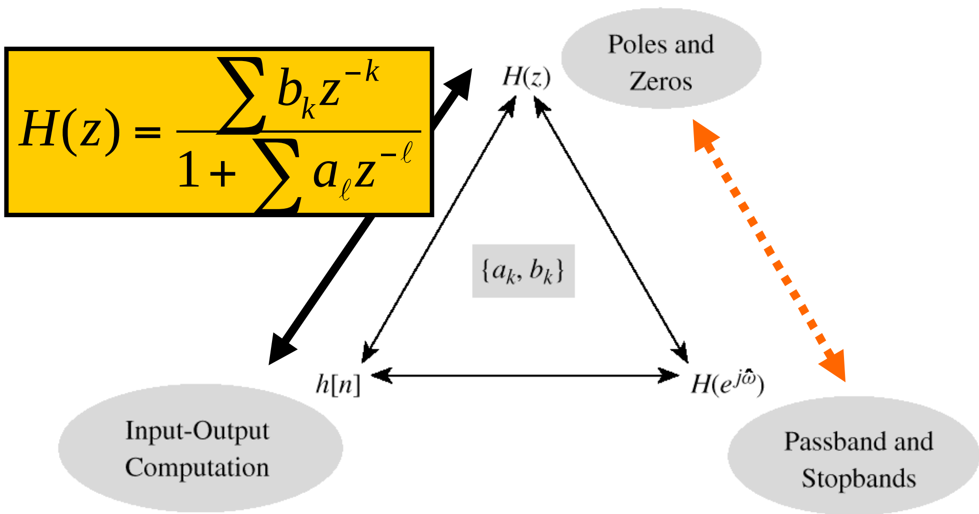
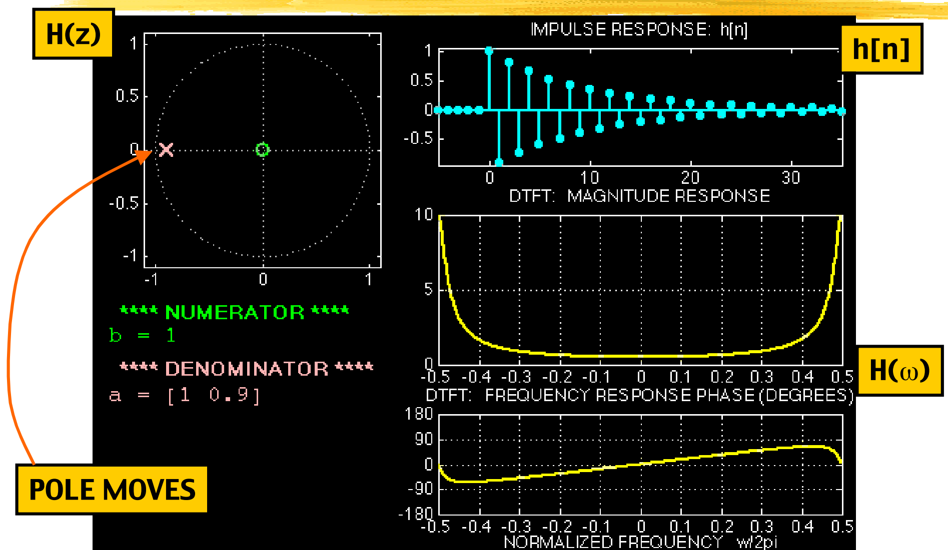


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k,\ b_k\}$ play a central role.

3 DOMAINS MOVIE: IIR



POP QUIZ

- **Given:** $H(z)=\dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- **Find the Impulse Response, h[n]**
- **Find the output, y[n]**
  - **When** $x[n]=\cos(0.25\pi n)$

SINUSOIDAL RESPONSE

- **x[n] = SINUSOID => y[n] is SINUSOID**
- **Get MAGNITUDE & PHASE from H(z)**

$$\text{if } x[n]=e^{j\hat{\omega}n}\text{, then}$$

$$y[n] = \mathcal{H}(\hat{\omega})e^{j\hat{\omega}n}$$

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

POP QUIZ ANSWER

- **Given:** $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- **The input:** $x[n] = \cos(0.25\pi n)$
- **Then y[n]** $y[n] = M\cos(0.25\pi n + \psi)$

$$H(e^{j0.25\pi}) = \frac{2 + 2e^{-j0.25\pi}}{1 - 0.8e^{-j0.25\pi}} = 5.18e^{-j0.417\pi}$$

POP QUIZ ANSWER: h[n]

- **Expand:** $\dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}} = \dfrac{2}{1 - 0.8z^{-1}} + \dfrac{2z^{-1}}{1 - 0.8z^{-1}}$
- **Find the Impulse Response, h[n]**
- **Use "INVERSE z−TRANSFORM"**

$$h[n] = 2(0.8)^n u[n] + 2(0.8)^{n-1} u[n-1]$$

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

SECOND-ORDER FILTERS

- **Two FEEDBACK TERMS**

SECOND-ORDER FILTERS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$(1 - a_1 z^{-1} - a_2 z^{-2}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

2nd ORDER EXAMPLE

Example 8.20

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - x[n-2]$$

$$y[n] - 0.9y[n-1] + 0.81y[n-2] = x[n] - x[n-2]$$

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HH = freqz( bb, aa, [-6:(pi/100):6] );
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aa = [ 1, −0.9, 0.81 ]     bb = [ 1, 0, −1 ]
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$$H(z) = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO REAL POLES

- Find Impulse Response ?
- Express H(z) as a SUM
  - Use PARTIAL FRACTIONS
  - Invert each term separately
    - with Z-Transform Table

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}}$$

EX: Partial Fractions

Example 8.10

$$X(z) = \frac{1 - 2.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}} = \frac{1 - 2.1z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})}$$

$$X(z) = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.8z^{-1}}$$

$$A = X(z)(1 + 0.5z^{-1})\Big|_{z=-0.5} = \frac{1 - 2.1z^{-1}}{1 - 0.8z^{-1}}\Bigg|_{z=-0.5} = \frac{1 + 4.2}{1 + 1.6} = 2$$

$$B = X(z)(1 - 0.8z^{-1})\Big|_{z=0.8} = \frac{1 - 2.1z^{-1}}{1 + 0.5z^{-1}}\Bigg|_{z=0.8} = \frac{1 - 2.1/0.8}{1 + 0.5/0.8} = -1$$

EX: Partial Fractions - 2

$$X(z) = \frac{2}{1 + 0.5z^{-1}} - \frac{1}{1 - 0.8z^{-1}}$$

$$x[n] = 2(-0.5)^n u[n] - (0.8)^n u[n]$$

Note that the poles at $z = p_1 = -0.5$ and $z = p_2 = 0.8$ give rise to terms in $x[n]$ of the form p_k^n.

GENERAL INVERSE Z

PROCEDURE FOR INVERSE z-TRANSFORMATION ($M < N$)

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.

2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\big|_{z=p_k}$$

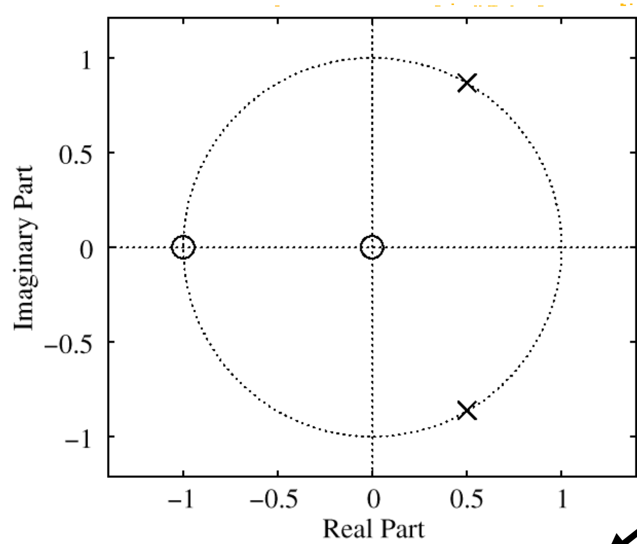
3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
- Find FREQUENCY RESPONSE
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make BANDPASS FILTER
    - Called “RESONANCE”

Complex POLE-ZERO PLOT

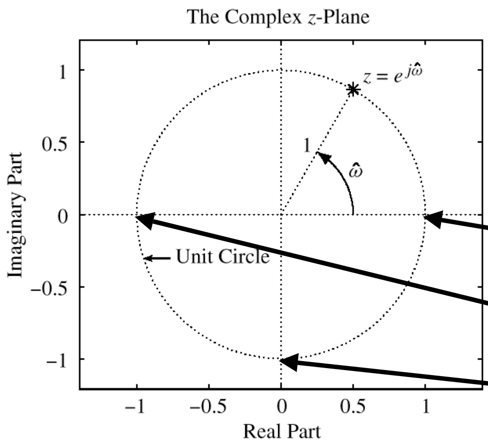


$$\frac{2z(z+1)}{z^2 - z + 1}$$

Pole-zero plot for system with $H(z) = \dfrac{2 + 2z^{-1}}{1 - z^{-1} + z^{-2}}$.

UNIT CIRCLE

- **MAPPING BETWEEN** z and $\hat{\omega}$

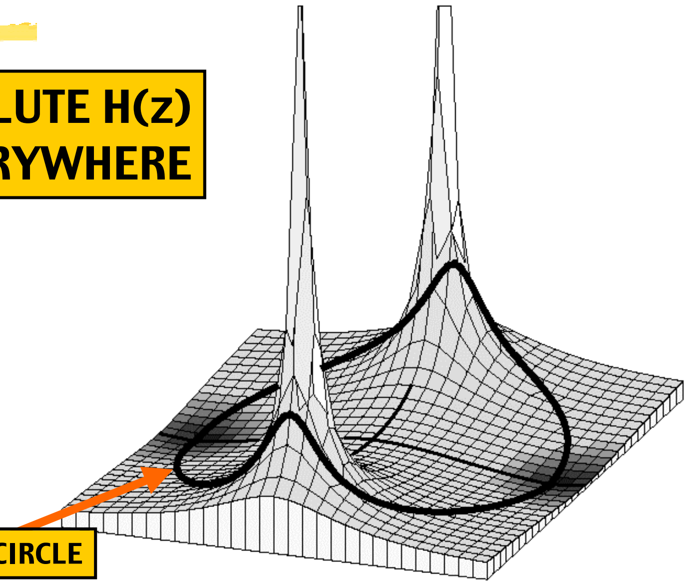


$$z = e^{j\hat{\omega}}$$

$z = 1$	$\leftrightarrow$	$\hat{\omega} = 0$
$z = -1$	$\leftrightarrow$	$\hat{\omega} = \pm\pi$
$z = \pm j$	$\leftrightarrow$	$\hat{\omega} = \pm\frac{1}{2}\pi$

3-D VIEW

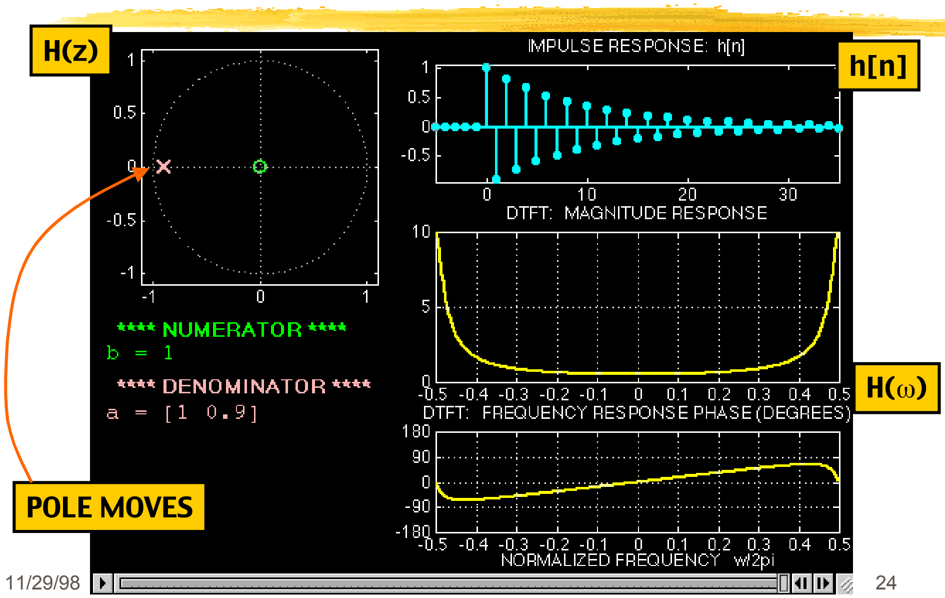
EVALUTE H(z) EVERYWHERE



UNIT CIRCLE

The poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.

3 DOMAINS MOVIE: IIR



THREE INPUTS

- **Given:**

$$\rightarrow \boxed{H(z) = \frac{1}{1 - 0.5z^{-1}}} \rightarrow$$

- **Find the output, y[n]**
  - **When**

$$x[n] = \cos(0.25\pi n)$$

$$x[n] = a^n u[n]$$

$$x[n] = \cos(0.25\pi n) u[n]$$

CONVOLUTION PROPERTY

$$x[n] = e^{j\hat{\omega}_0 n}u[n]$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

$H(z)$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

SPLIT Y(z)

- Need SUM of Terms:

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

$$= \frac{b_0}{(1 - a_1 z^{-1})(1 - e^{j\hat{\omega}_0}z^{-1})}$$

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

INVERT Y(z) to y[n]

- Use the Z-Transform Table

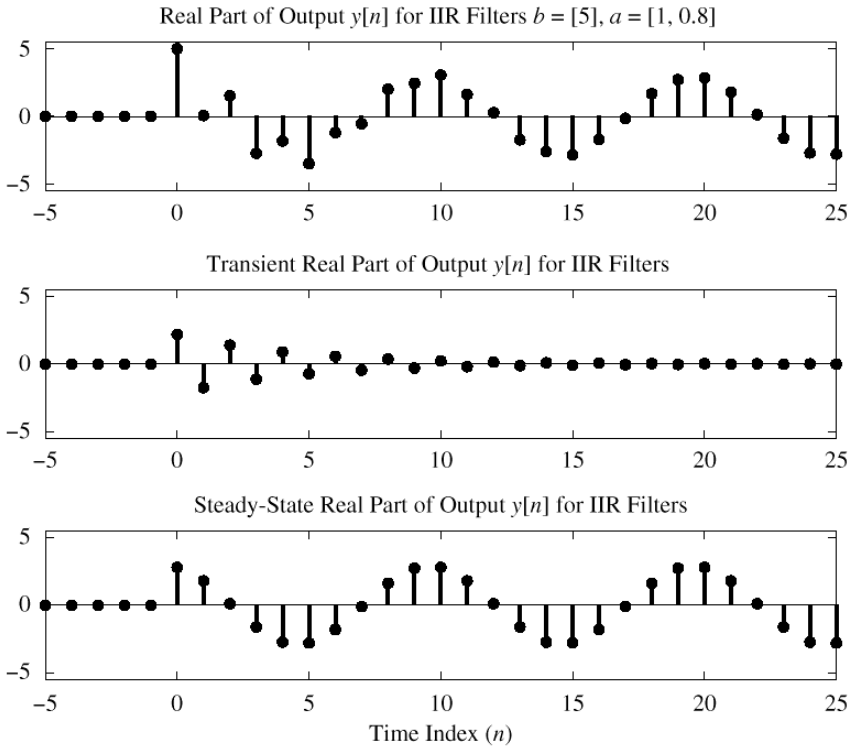
$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)e^{j\hat{\omega}_0 n}u[n]$$

TWO PARTS of y[n]

- TRANSIENT
  - Acts Like h[n]
  - Dies out ?
- STEADY-STATE
  - Depends on the input
  - e.g., Sinusoidal

$$h[n] = a^n u[n]$$



STEADY STATE

- When Transient dies out
- Limit as "n" approach infinity
- Use Frequency Reponse to get Magnitude & Phase for sinusoid

$$y[n] \rightarrow \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

STABILITY

- Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

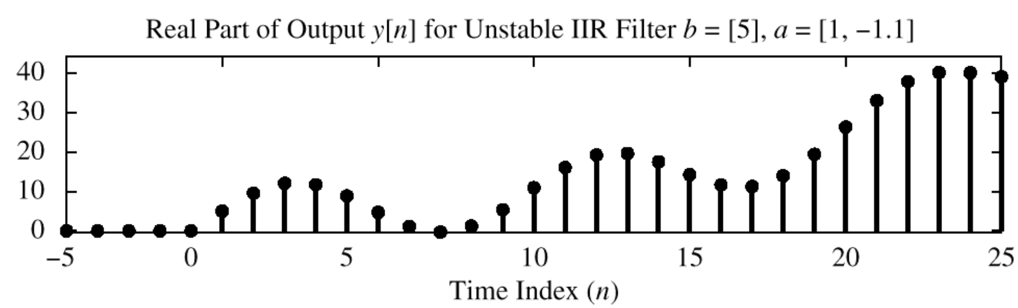


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.