EE-2200 Fall-98

Lecture 5

Fourier Series Coefficients

9-Oct-98

Web-CT Info

- Check the Bulletin Board for msgs
- Lectures are being posted
  - PDF format (4 per page)
- Calendar has entries:
  - Quiz #1 on 16-Oct (Friday)
  - Quiz #2 on 13-Nov (Friday)

Homework Info

- Prob Set #2 due Monday, 12-Oct
  - In Lecture, before NOON
- Solutions will be posted on Tues
  - HW #1 solutions are on the web site

Lab Info

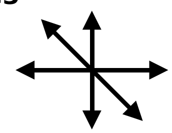
- Lab #2 Report
  - Turn in during your lab time
  - Write-up sections 4 and 5
  - Include INSTRUCTOR VERIFICATION
- Lab #3 will be posted late on Friday
  - Music Synthesis Lab (learn notation)

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, pp. 57–68
- Other Reading:
  - Notes on Fourier Series
    - (3 pages posted to WebCT)
  - Next Lecture: Chap. 3, pp. 68–77

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - (A_k, ω_k, ϕ_k)
- Plots & Sketches
  - x(t) versus t
  - Spectrum



- Recorded Signals
  - Speech
  - Music
  - No simple formula
- MATLAB
  - Numerical
  - Computation
  - Plotting lists of numbers

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

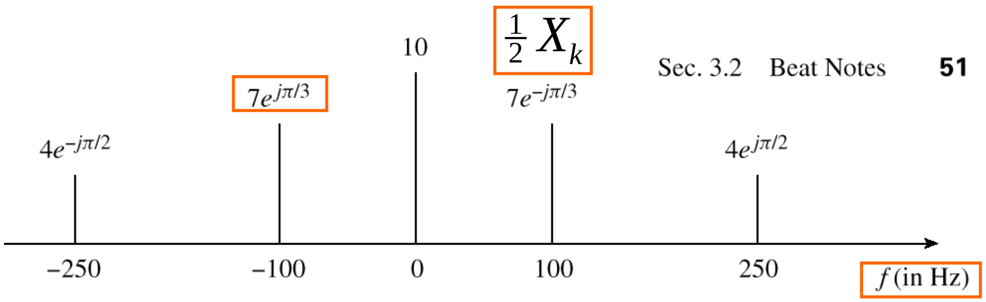
- ANALYSIS via Fourier Series
  - For PERIODIC signals: $x(t+T) = x(t)$

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

FREQUENCY DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\phi_k}$$

frequency is f_k.

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$x(t) = x(t+T)$$

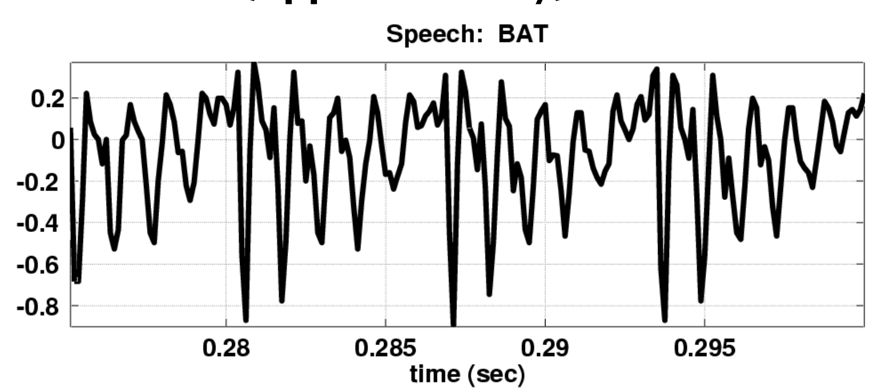
  - Example:

$$x(t) = \cos^2(3t)$$

  - Speech can be “quasi-periodic”

Speech Signal: BAT

- Nearly **Periodic** in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Period of Complex Exp

$$x(t) = e^{j\omega_0 t}$$

$$x(t+T) = x(t)\,?$$

Period is T

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega_0 T} = 1 \Rightarrow \omega_0 T = 2\pi k$$

$$\omega_0 = \tfrac{2\pi k}{T} = (\tfrac{2\pi}{T})k$$

k = integer

HARMONIC SIGNAL

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k\, f_0$$

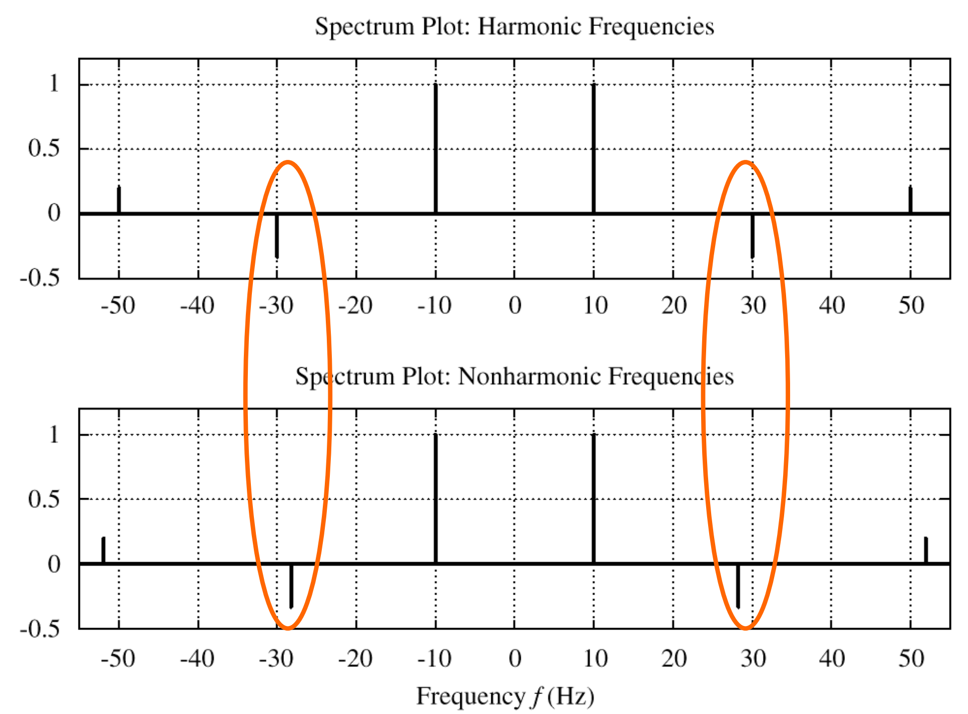
$$f_0 = \textit{fundamental frequency}$$

Harmonic Signal Spectrum

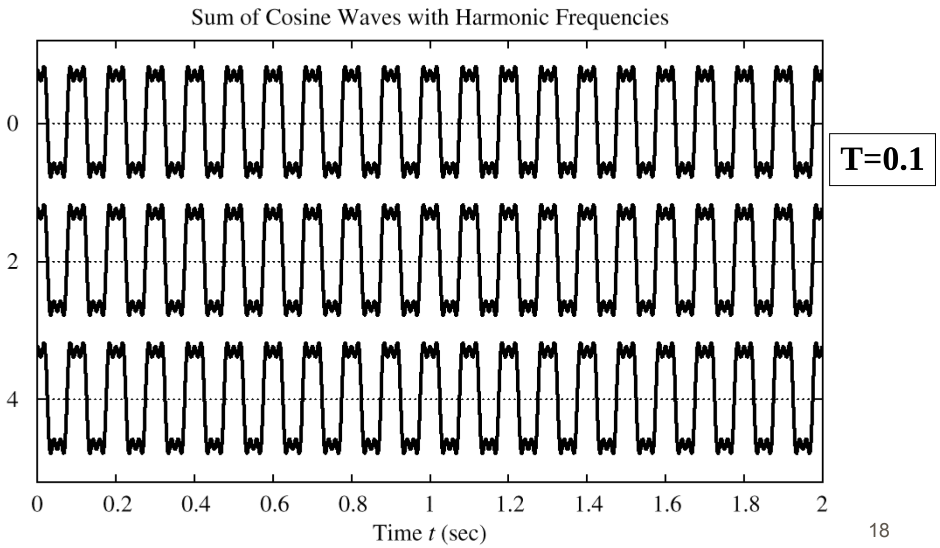
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

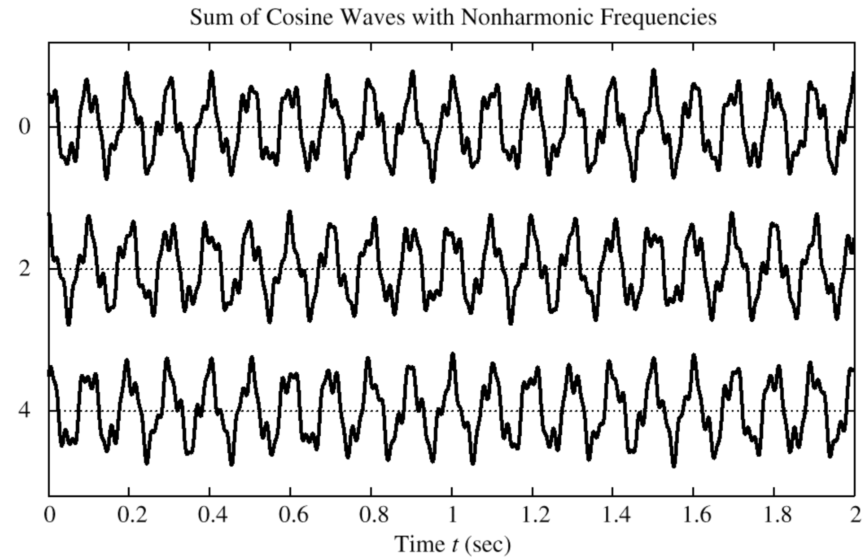
$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$



Harmonic Signal (3 Freqs)

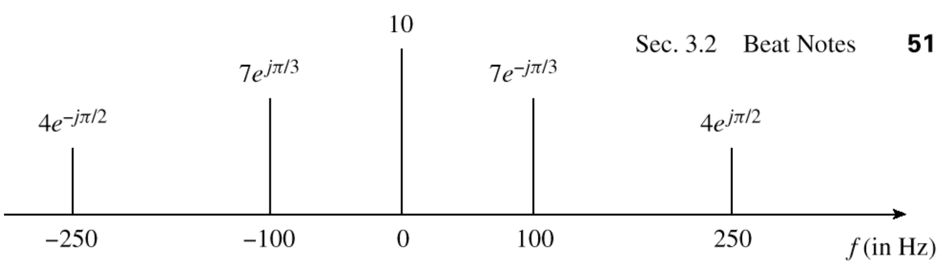


NON-Harmonic Signal



PERIOD from SPECTRUM

- Add the spectrum components:



What is the PERIOD for the signal x(t)?

Multiples of Fundamental Frequency

Frequencies:		Amplitude & Phase	
−250 Hz	(k=−5)	4	$-\pi/2$
−100 Hz	(k=−2)	7	$+\pi/3$
0 Hz	(k=0)	10	0
100 Hz	(k=+2)	7	$-\pi/3$
250 Hz	(k=+5)	4	$+\pi/2$

FUNDAMENTAL FREQUENCY = 50 Hz
How do you find f_0?
Determine GCD: Greatest Common Divisor

Example: Synthetic Vowel

- What is the Fundamental Frequency ?

f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	−0.185
1600	$(1657 - j13520)$	13,621	−1.449
1700	$4723 + j0$	4723	0

Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

Vowel Waveform (sum of all 5 components)

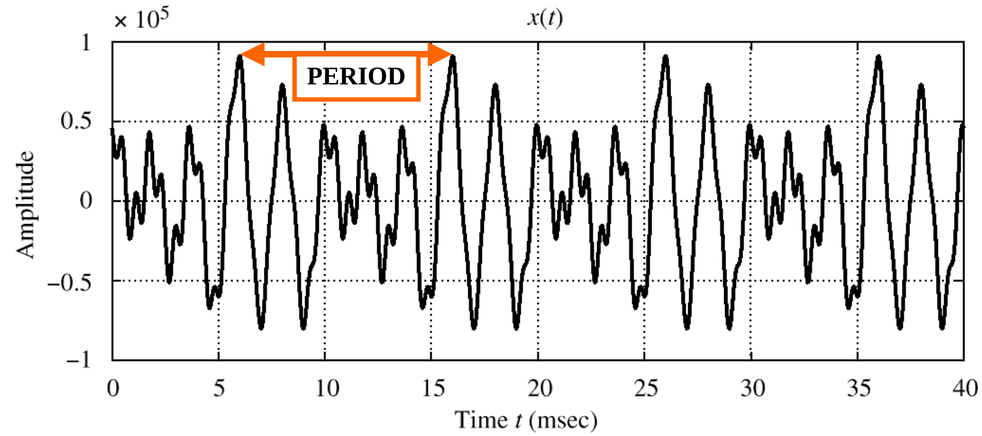


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - Easy
  - Given (ω_k, A_k, ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - Hard
  - Given x(t), extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

Fourier Series Expansion

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_{-k} e^{-j2\pi k f_0 t}$$

$$X_{-k} = X_k^* \quad \text{when } x(t) \text{ is real}$$

Fourier Series Integral

- **Determine X_k from x(t)**

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt$$

$f_0 = 1/T_0$

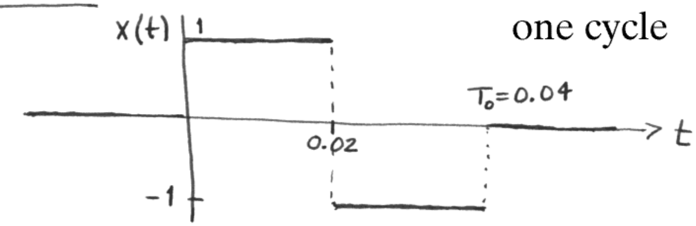
$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

SQUARE WAVE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ -1 & \frac{1}{2}T_0 \le t < T_0 \end{cases} \qquad (3.4.4)$$

Draw a plot of the square wave defined in (3.4.4) for $T_0 = 0.04$ sec.

Ex 3.3



FS for a SQUARE WAVE

$$X_k = \frac{2}{T_0}\int_0^{\frac{1}{2}T_0}(1)e^{-j2\pi kt/T_0}\,dt + \frac{2}{T_0}\int_{\frac{1}{2}T_0}^{T_0}(-1)e^{-j2\pi kt/T_0}\,dt$$

which can be manipulated as follows:[5]

$$\begin{aligned} X_k &= \frac{2}{T_0}\frac{e^{-j2\pi k(\frac{1}{2}T_0)/T_0} - e^{-j2\pi k(0)/T_0}}{-j2\pi k/T_0} + \frac{(-2)}{T_0}\frac{(e^{-j2\pi kT_0/T_0} - e^{-j2\pi k(\frac{1}{2}T_0)/T_0})}{-j2\pi k/T_0} \\ &= \frac{e^{-j\pi k} - 1}{-j\pi k} + \frac{e^{-j\pi k} - e^{-j2\pi k}}{-j\pi k} \\ &= \frac{2 - 2e^{-j\pi k}}{j\pi k} = \frac{2(1-(-1)^k)}{j\pi k} \end{aligned}$$

[5] We use the fact that $e^{-j2\pi k} = 1$ when k is an integer.

observe that the average value of this signal is zero, so $X_0 = 0$.

$$X_k = \begin{cases} \dfrac{4}{j\pi k} & k = \pm1, \pm3, \pm5, \ldots \\ 0 & k = 0, \pm2, \pm4, \pm6, \ldots \end{cases} \qquad (3.4.5)$$

The magnitude of these coefficients is shown in Fig. 3.12. The phase angles are $-\pi/2$ for $k > 0$, and $\pi/2$ for $k < 0$. Note that if $f_0 = 1/T_0 = 25$ Hz, only the frequencies at ±25, ±75, ±125, etc. are in the spectrum.

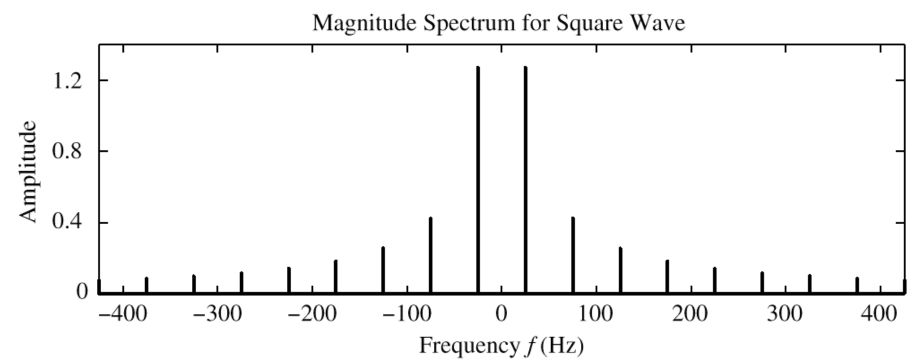
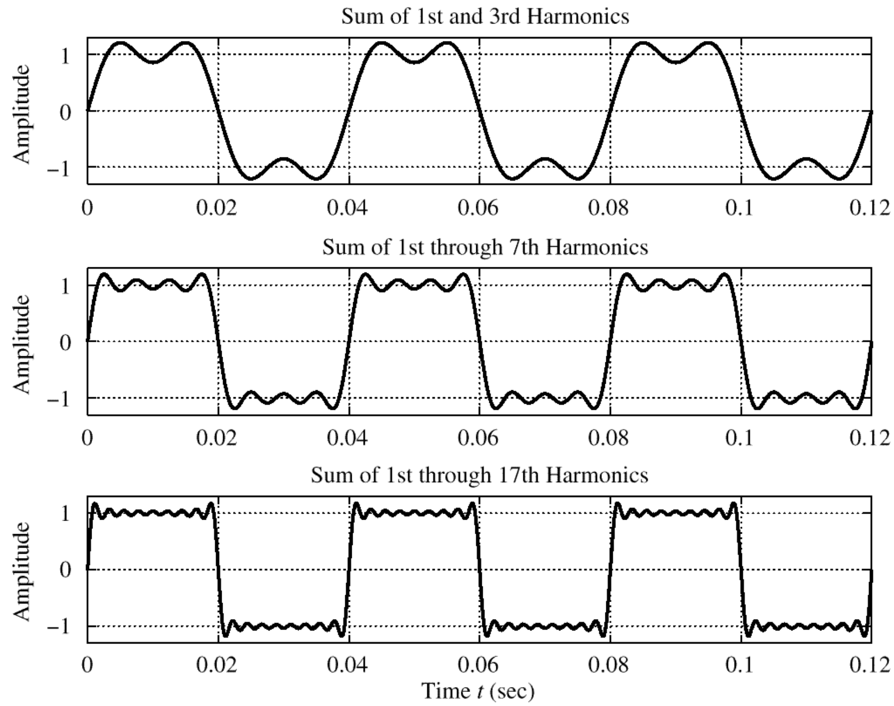
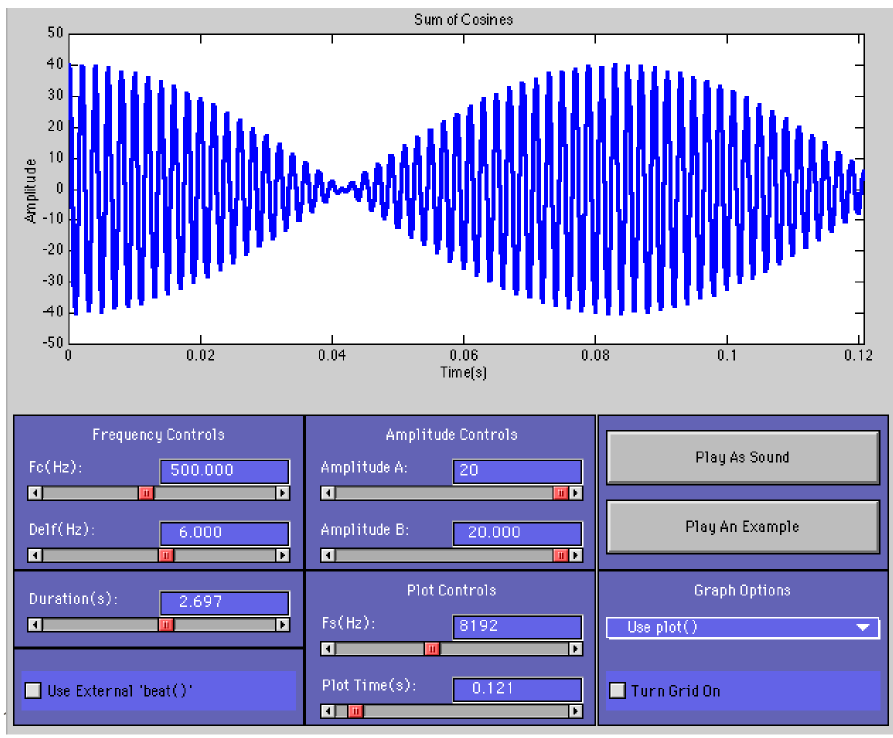


Figure 3.12 Spectrum of the square-wave signal whose Fourier series coefficients are given in (3.4.5) with $f_0 = 1/T_0 = 25$ Hz.

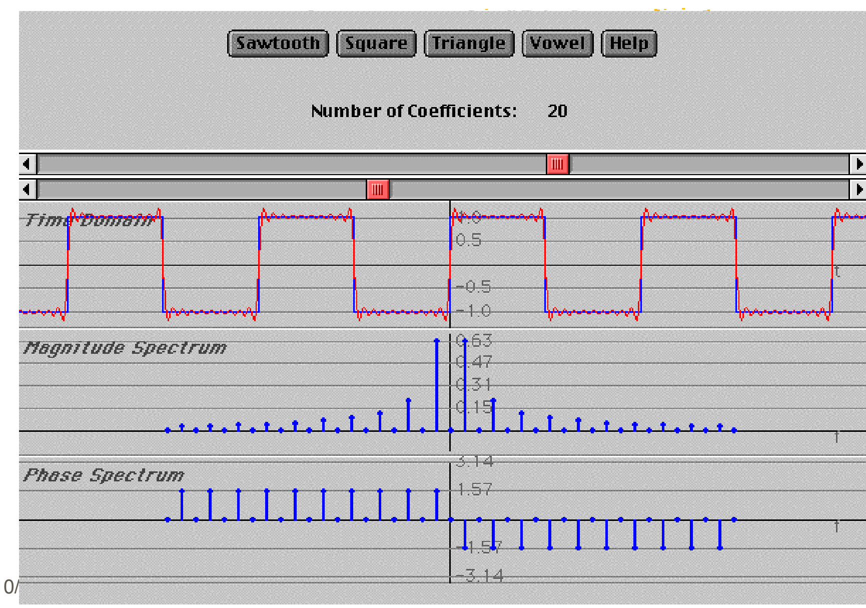


A Couple of DEMOS

- **Beat Control GUI**
  - **DSPFirst Toolbox: MATLAB**
    - **DSPFIRST/beatcon.m**
- **Fourier Series Java Applet**
  - **Interactive**
  - **http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html**



Fourier Series Java Applet



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case