EE-2200 Winter-99

Lecture 5

Fourier Series Coefficients

25-Jan-99

Web-CT Info

- Check the Bulletin Board for msgs
- Lectures are being posted
- Old Quizzes & Problems are linked
- Quiz #1 on 1–Feb (Monday)
  - In the ECE Auditorium
  - Alternate Time: 4pm Saturday 30–Jan

Homework Info

- Prob Set #3 due Friday, 29–Jan
  - In Lecture, before 2–PM
- Solutions will be posted on Friday
- Prob Set #4 will begin on 5–Feb

Lab Info

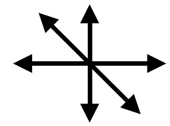
- Lab #2 Report
  - Turn in during your lab time
  - Write–up sections 4 and 5
  - Include INSTRUCTOR VERIFICATION
- Lab #3 was posted on Sunday
  - Music Notation will be needed

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, pp. 57–68
- Other Reading:
  - Notes on Fourier Series
    - (3 pages posted to WebCT)
  - Next Lecture: Chap. 3, pp. 68–77

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - (A_k, ω_k, ϕ_k)
- Plots & Sketches
  - x(t) versus t
  - Spectrum



- Recorded Signals
  - Speech
  - Music
  - No simple formula
- MATLAB
  - Numerical
  - Computation
  - Plotting lists of numbers

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

- ANALYSIS via Fourier Series
  - For PERIODIC signals: $x(t+T) = x(t)$

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

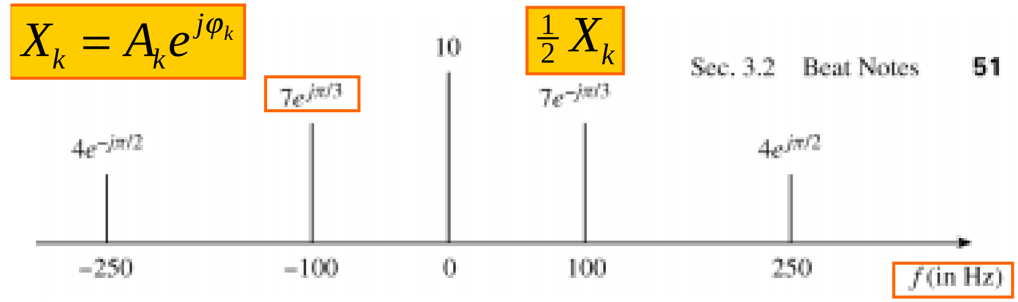
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

FREQUENCY DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\phi_k}$$

frequency is f_k.

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$x(t) = x(t+T)$$

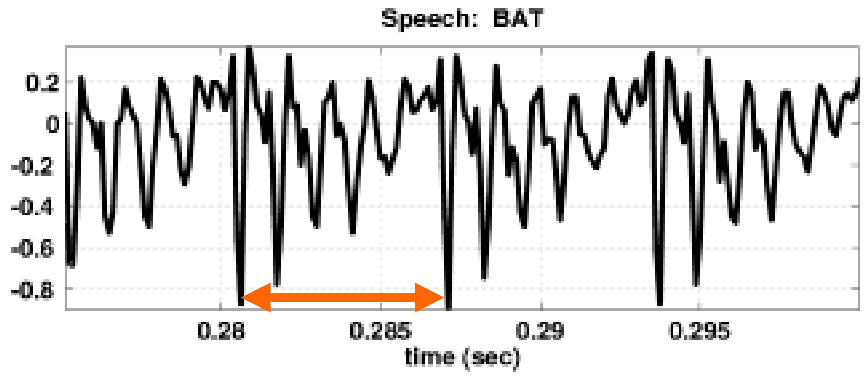
  - Example:

$$x(t) = \cos^2(3t) \qquad T = ?$$

  - Speech can be “quasi-periodic”

Speech Signal: BAT

- **Nearly Periodic in the Vowel Region**
  - **Period is (Approximately) T = 0.0065 sec**



Period of Complex Exp

$$x(t) = e^{j\omega_0 t}$$

$$x(t+T) = x(t)\ ?$$ **Period is T**

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega_0 T} = 1 \quad \Rightarrow \omega_0 T = 2\pi k$$

$$\omega_0 = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k$$ **k = integer**

HARMONIC SIGNAL

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k\, f_0$$

$$f_0 = \text{fundamental frequency}$$

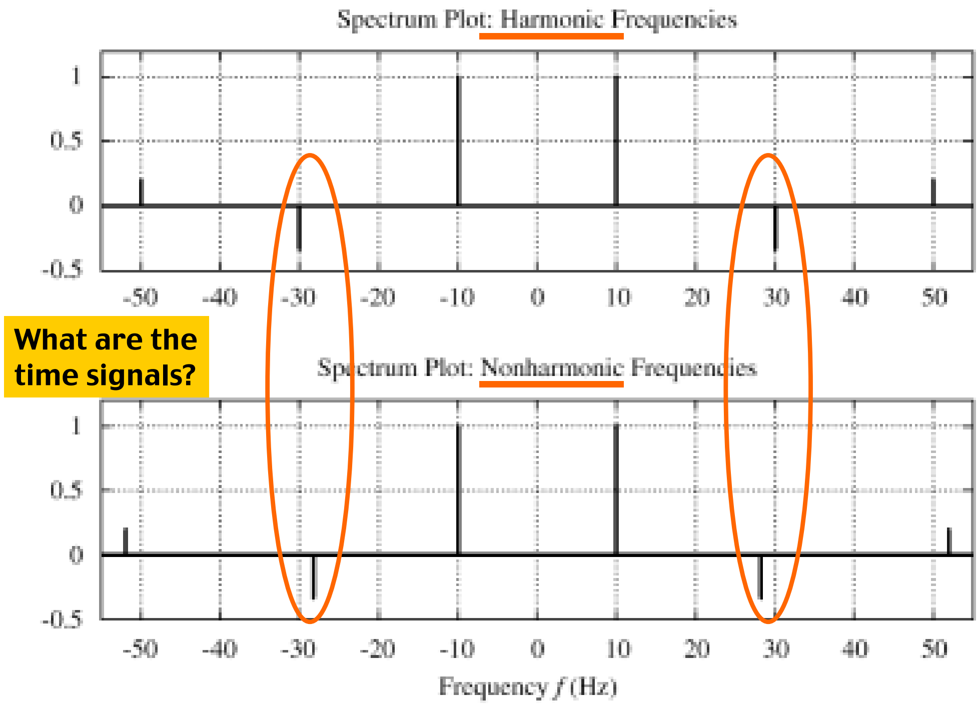
$$T_0 = \text{fundamental Period}$$

Harmonic Signal Spectrum

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

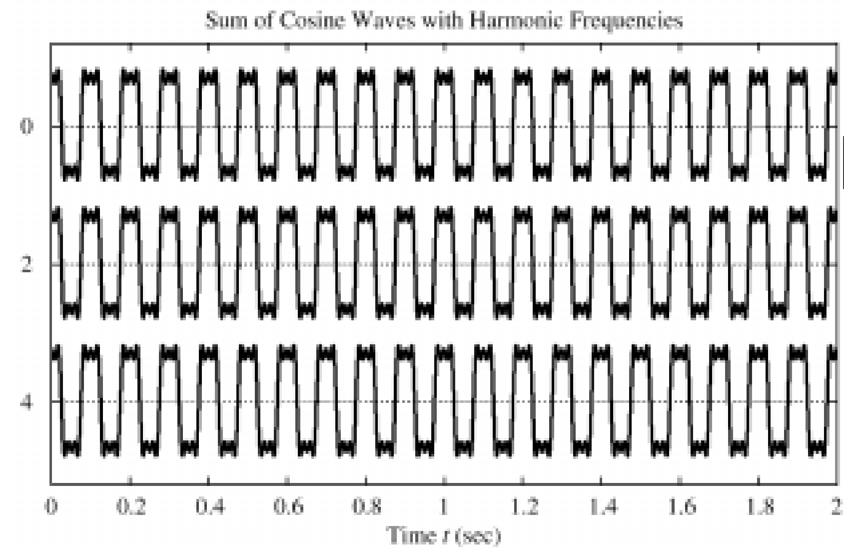
$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$



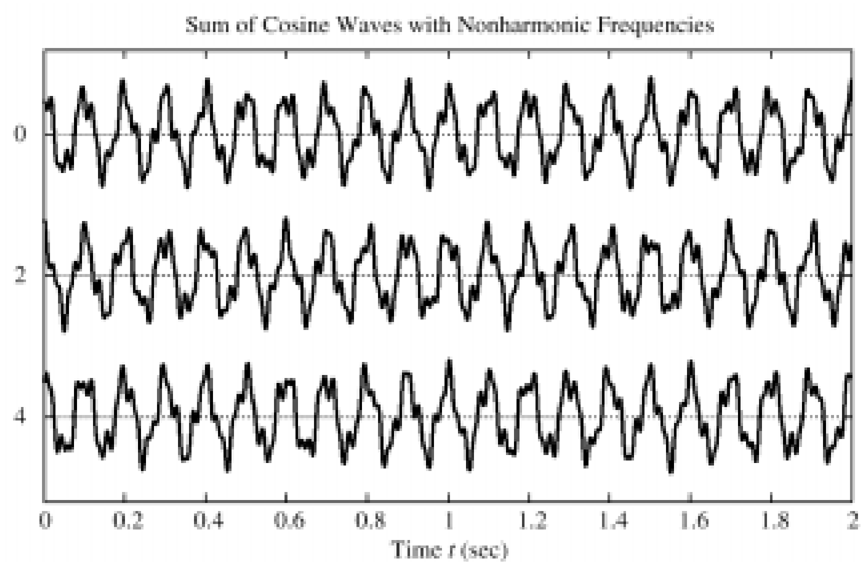
What are the time signals?

Harmonic Signal (3 Freqs)



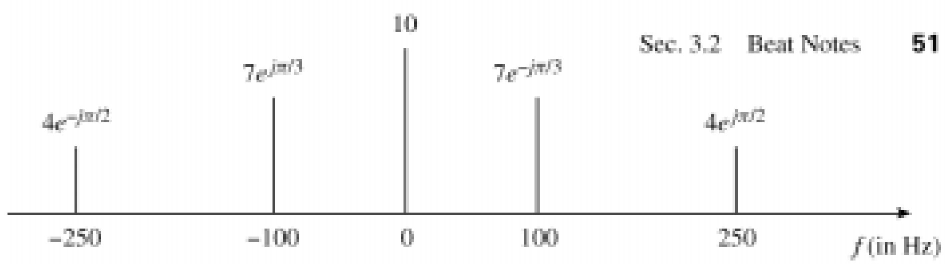
T=0.1

NON-Harmonic Signal



PERIOD from SPECTRUM

- Add the spectrum components:



What is the PERIOD for the signal x(t)?

Multiples of Fundamental Frequency

- Frequencies:
  - −250 Hz (k=−5)
  - −100 Hz (k=−2)
  - 0 Hz (k=0)
  - 100 Hz (k=+2)
  - 250 Hz (k=+5)
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

FUNDAMENTAL FREQUENCY = 50 Hz
How do you find f_0 ?
Determine GCD: Greatest Common Divisor

Example: Synthetic Vowel

- What is the Fundamental Frequency ?

f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	−0.185
1600	$(1657 - j13520)$	13,621	−1.449
1700	$4723 + j0$	4723	0

Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

Vowel Waveform (sum of all 5 components)

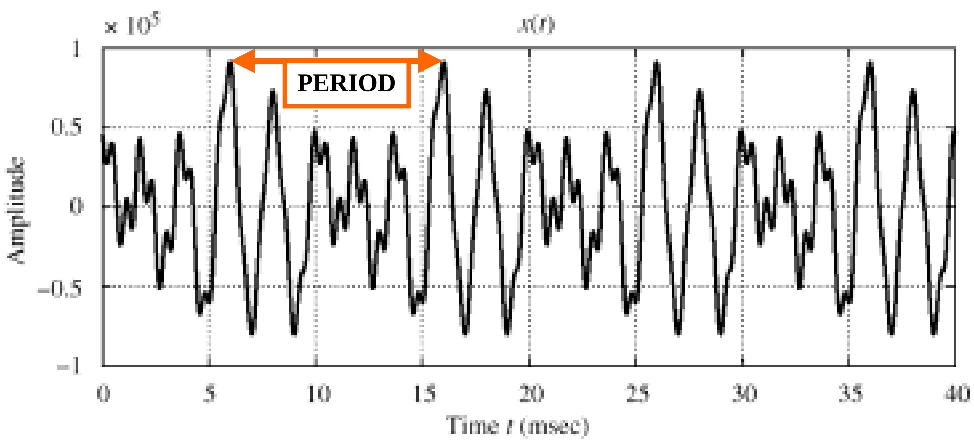


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - Easy
  - Given (ω_k, A_k, ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - Hard
  - Given x(t), extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

Fourier Series Expansion

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$\boxed{X_k = A_k e^{j\varphi_k}}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_{-k} e^{-j2\pi k f_0 t}$$

$$X_{-k} = X_k^* \quad \text{when } x(t) \text{ is real}$$

Fourier Series Integral

- **Determine X_k from x(t)**

$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k t / T_0}\, dt$$

$f_0 = 1/T_0$

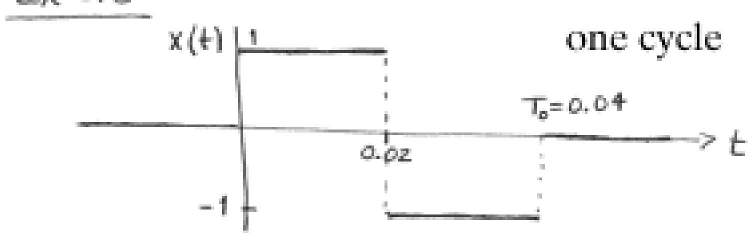
$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$$

SQUARE WAVE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ -1 & \frac{1}{2}T_0 \le t < T_0 \end{cases} \qquad (3.4.4)$$

Draw a plot of the square wave defined in (3.4.4) for $T_0 = 0.04$ sec.

Ex 3.3



FS for a SQUARE WAVE

$$X_k = \frac{2}{T_0}\int_0^{\frac{1}{2}T_0} (1) e^{-j2\pi k t/T_0}\, dt + \frac{2}{T_0}\int_{\frac{1}{2}T_0}^{T_0} (-1) e^{-j2\pi k t/T_0}\, dt$$

which can be manipulated as follows:[5]

$$\begin{aligned} X_k &= \frac{2}{T_0}\,\frac{e^{-j2\pi k(\frac{1}{2}T_0)/T_0} - e^{-j2\pi k(0)/T_0}}{-j2\pi k/T_0} + \frac{(-2)}{T_0}\,\frac{(e^{-j2\pi k T_0/T_0} - e^{-j2\pi k(\frac{1}{2}T_0)/T_0})}{-j2\pi k/T_0} \\ &= \frac{e^{-j\pi k} - 1}{-j\pi k} + \frac{e^{-j\pi k} - e^{-j2\pi k}}{-j\pi k} \\ &= \frac{2 - 2e^{-j\pi k}}{j\pi k} = \frac{2(1 - (-1)^k)}{j\pi k} \end{aligned}$$

[5] We use the fact that $e^{-j2\pi k} = 1$ when k is an integer.

observe that the average value of this signal is zero, so $X_0 = 0$.

$$X_k = \begin{cases} \dfrac{4}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \pm 6, \ldots \end{cases} \qquad (3.4.5)$$

The magnitude of these coefficients is shown in Fig. 3.12. The phase angles are $-\pi/2$ for $k > 0$, and $\pi/2$ for $k < 0$. Note that if $f_0 = 1/T_0 = 25$ Hz, only the frequencies at ± 25, ± 75, ± 125, etc. are in the spectrum.

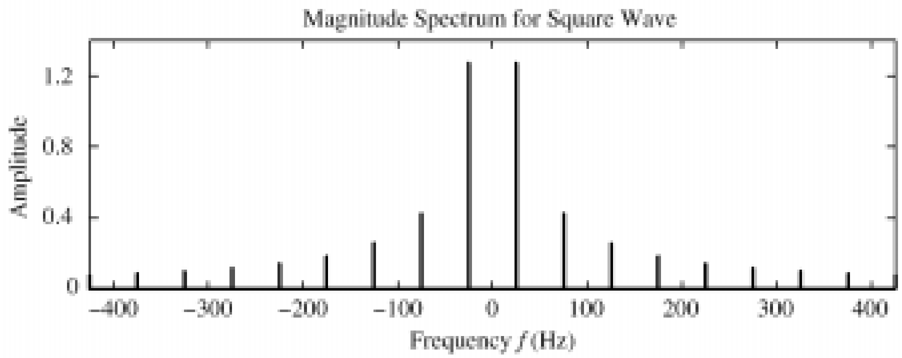
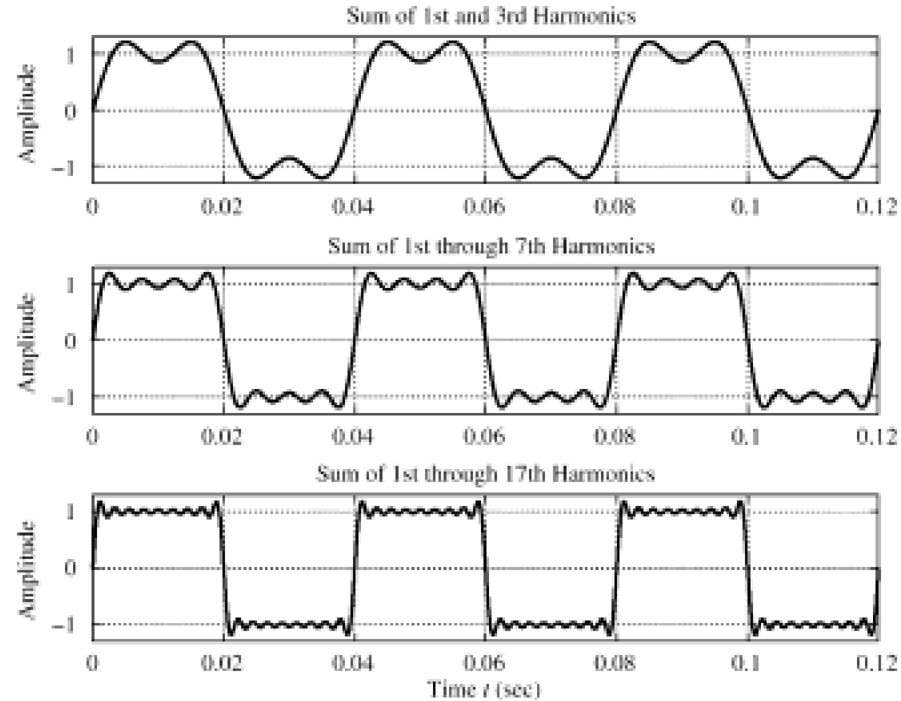


Figure 3.12 Spectrum of the square-wave signal whose Fourier series coefficients are given in (3.4.5) with $f_0 = 1/T_0 = 25$ Hz.



A Couple of DEMOS

- **Beat Control GUI**
  - **DSPFirst Toolbox: MATLAB**
    - **DSPFIRST/beatcon.m**
- **Fourier Series Java Applet**
  - **Interactive**
  - **http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html**

Fourier Series Java Applet