EE-2025 Fall-99

Lecture 16

2nd-ORDER SYSTEMS

29-Oct-99

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 22-Nov
- Get NEW CHAPTERS next week
  - PDF or Bookstore
- Prob Set #9 is due Friday
- Lab #9 on Pole-Zero Editor (PeZ)
  - Then 3 more Labs

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 279-300
- Other Reading:
  - Recitation: Ch. 8, pp. 261-272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, all

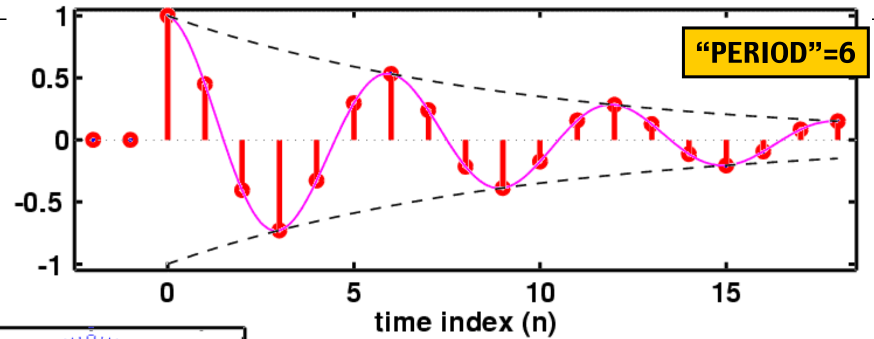
LECTURE OBJECTIVES

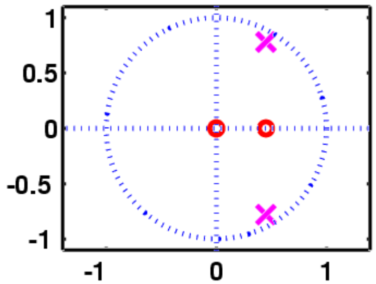
- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS
- INVERSE z-TRANSFORM
- h[n] can OSCILLATE

h[n]: Decays & Oscillates





$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

THREE DOMAINS

$$H(z) = \frac{\sum b_k z^{-k}}{1 - \sum a_\ell z^{-\ell}}$$

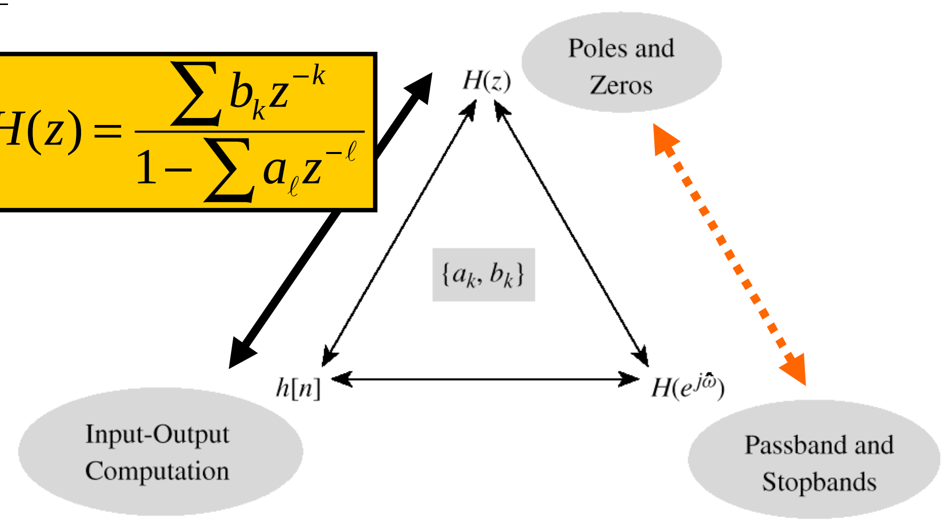


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k, b_k\}$ play a central role.

FIRST-ORDER H(z)

- H(z) = z−Transform of h[n]

$$y[n] = ay[n-1] + x[n]$$

$$h[n] = a^n u[n]$$

POLE @ z=a

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

POP QUIZ

- Given: $H(z) = \dfrac{2 - 2z^{-1}}{1 + 0.8z^{-1}}$
- Find the STEP RESPONSE
  - Find the output y[n] when the input is

$$x[n] = u[n] \qquad X(z) = \frac{1}{1 - z^{-1}}$$

PQ ANSWER: STEP RESPONSE

$$Y(z) = H(z)X(z) = \frac{2 - 2z^{-1}}{1 + 0.8z^{-1}} \times \frac{1}{1 - z^{-1}}$$

- Simplify the algebra and then
- Use "INVERSE z-TRANSFORM"

$$Y(z) = \frac{2}{1 + 0.8z^{-1}} \quad \Rightarrow \quad y[n] = 2(-0.8)^n u[n]$$

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\frac{1}{1 - az^{-1}}$

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

SECOND-ORDER FILTERS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$(1 - a_1 z^{-1} - a_2 z^{-2}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

2nd ORDER EXAMPLE

Example 8.20

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - x[n-2]$$

$$y[n] - 0.9y[n-1] + 0.81y[n-2] = x[n] - x[n-2]$$

```
HH = freqz( bb, aa, [-6:(pi/100):6] );

aa = [ 1, −0.9, 0.81 ]    bb = [ 1, 0, −1 ]
```

$$H(z) = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO REAL POLES

- Find Impulse Response ?
- Express H(z) as a SUM
  - Use PARTIAL FRACTIONS
  - Invert each term separately
    - with Z-Transform Table

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}}$$

EX: Partial Fractions

Example 8.10

$$X(z) = \frac{1 - 2.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}} = \frac{1 - 2.1z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})}$$

$$X(z) = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.8z^{-1}}$$

$$A = X(z)(1 + 0.5z^{-1})\Big|_{z=-0.5} = \frac{1 - 2.1z^{-1}}{1 - 0.8z^{-1}}\Bigg|_{z=-0.5} = \frac{1 + 4.2}{1 + 1.6} = 2$$

$$B = X(z)(1 - 0.8z^{-1})\Big|_{z=0.8} = \frac{1 - 2.1z^{-1}}{1 + 0.5z^{-1}}\Bigg|_{z=0.8} = \frac{1 - 2.1/0.8}{1 + 0.5/0.8} = -1$$

EX: Partial Fractions - 2

$$X(z) = \frac{2}{1 + 0.5z^{-1}} + \frac{-1}{1 - 0.8z^{-1}}$$

$$x[n] = 2(-\tfrac{1}{2})^n u[n] - (0.8)^n u[n]$$

Note that the poles at $z = p_1 = -0.5$ and $z = p_2 = 0.8$ give rise to terms in $x[n]$ of the form p_k^n.

$$(p_k)^n u[n]$$

GENERAL INVERSE Z

PROCEDURE FOR INVERSE z-TRANSFORMATION ($M < N$)

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

(pole)^n

TWO COMPLEX POLES

- **Find Impulse Response ?**
  - **Can OSCILLATE vs. n**
  - **"RESONANCE"**

$$(p_k)^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

- **Find FREQUENCY RESPONSE**
  - **Depends on Pole Location**
  - **Close to the Unit Circle?**
    - **Make BANDPASS FILTER**

2nd ORDER EXAMPLE

$$h[n] = 0.9^n \cos(\tfrac{\pi}{3} n) u[n] = 0.9^n \tfrac{1}{2}\left(e^{j\pi n/3} + e^{-j\pi n/3}\right) u[n]$$

$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\tfrac{\pi}{3}) z^{-1}}{(1 - 0.9e^{j\pi/3} z^{-1})(1 - 0.9e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

2nd ORDER Z-transform PAIR

$$h[n] = r^n \cos(\theta n) u[n] = r^n \tfrac{1}{2}\left(e^{j\theta n} + e^{-j\theta n}\right) u[n]$$

$$H(z) = \frac{0.5}{1 - re^{j\theta} z^{-1}} + \frac{0.5}{1 - re^{-j\theta} z^{-1}}$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

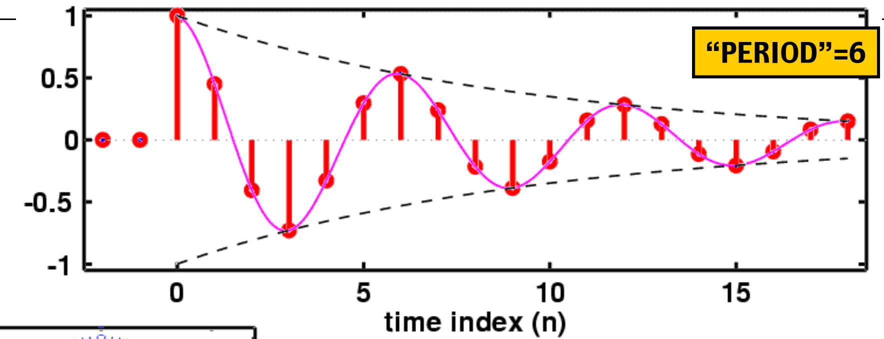
2nd ORDER Z-transform: sine

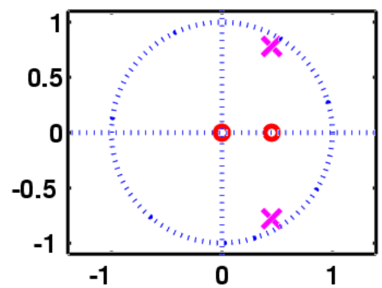
$$h[n] = r^n \sin(\theta n) u[n] = r^n \tfrac{1}{2j}\left(e^{j\theta n} - e^{-j\theta n}\right) u[n]$$

$$H(z) = -\frac{0.5j}{1 - re^{j\theta}z^{-1}} + \frac{0.5j}{1 - re^{-j\theta}z^{-1}}$$

$$H(z) = \frac{-r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

h[n]: Decays & Oscillates

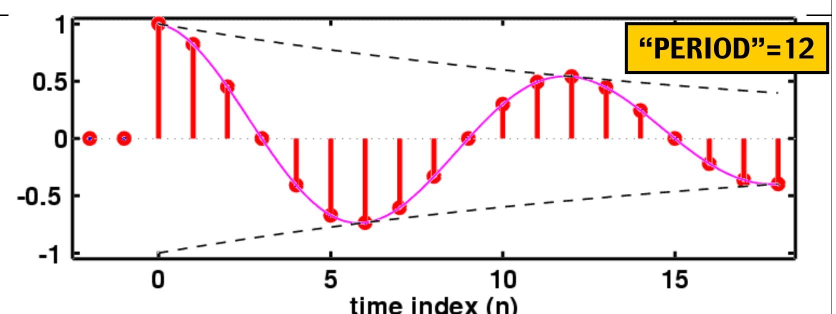


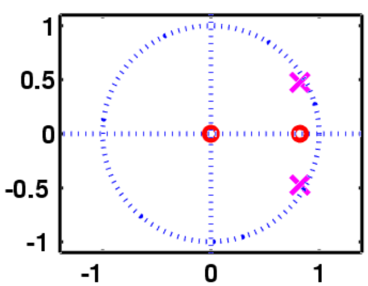


$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

h[n]: Decays & Oscillates





$$h[n] = (0.95)^n \cos(\tfrac{\pi}{6} n) u[n]$$

$$\frac{1 - 0.8227z^{-1}}{1 - 1.6454z^{-1} + 0.9025z^{-2}}$$

3 DOMAINS MOVIE: IIR