EE-2025 Fall-99

Lecture 17

3-Domains for IIR

1-Nov-99

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 22-Nov
- Get NEW CHAPTERS
  - Web-CT (PDF), or Bookstore
- Prob Set #10 is due Friday
- Lab #9 on Pole-Zero Editor (PeZ)
  - Then 3 more Labs

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 279-300
- Other Reading:
  - Recitation: Ch. 8, pp. 261-272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, all

LECTURE OBJECTIVES

- Z-TRANSFORM METHOD
  - PARTIAL FRACTIONS
  - 2nd-ORDER FILTERS, OSCILLATING INPUTS

$$Y(z) = H(z)X(z)$$

- STABILITY
- THREE-DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

THREE DOMAINS

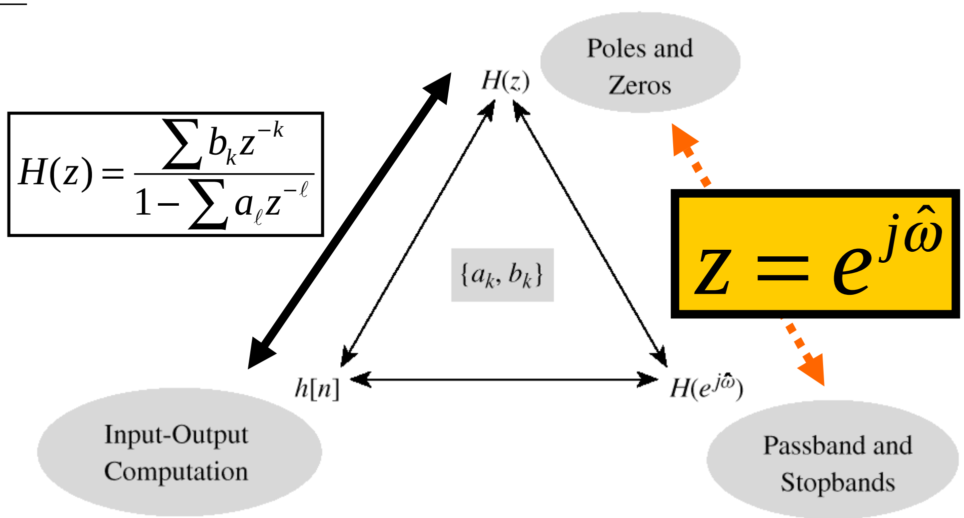


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k, b_k\}$ play a central role.

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\frac{1}{1 - az^{-1}}$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

2nd ORDER Z-transform PAIR

$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = Ar^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = A\frac{\cos\varphi - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

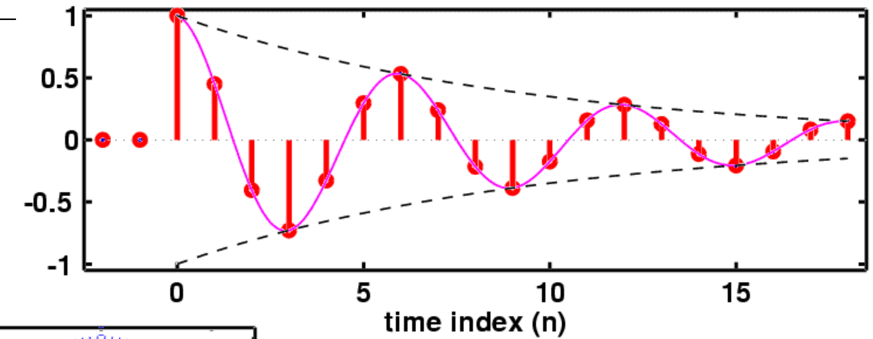
2nd ORDER EX: n-Domain

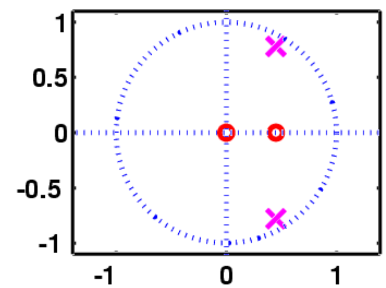
$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

$$y[n]=0.9y[n-1]-0.81y[n-2]+x[n]-0.45x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```


h[n]: Decays & Oscillates





$$h[n]=(0.9)^n\cos(\tfrac{\pi}{3}n)u[n]$$

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

GENERAL INVERSE Z

PROCEDURE FOR INVERSE z-TRANSFORMATION $(M < N)$

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1-p_k z^{-1})$ for $k=1,2,\ldots,N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1-p_k z^{-1})\big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

(pole)^n

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

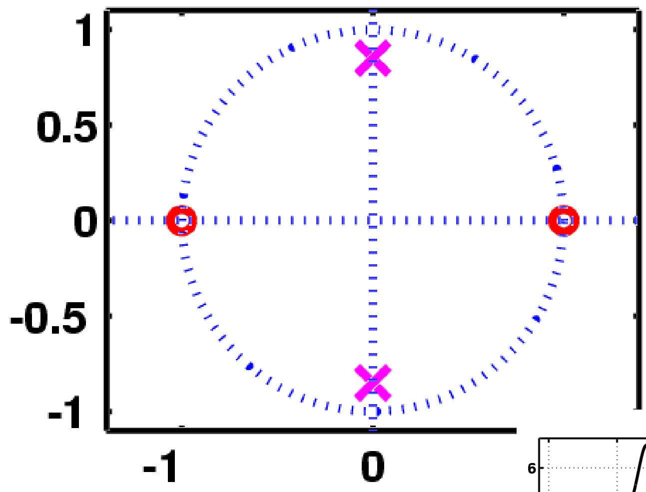
$$(p_k)^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

- Find <u>FREQUENCY RESPONSE</u>
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make <u>BANDPASS FILTER</u>

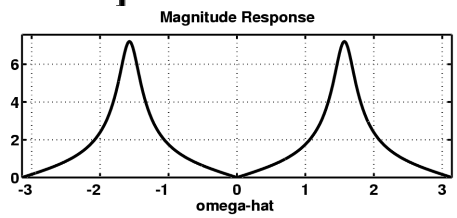
$$\text{pole} = re^{j\theta}$$

$$r \to 1\,?$$

Complex POLE-ZERO PLOT

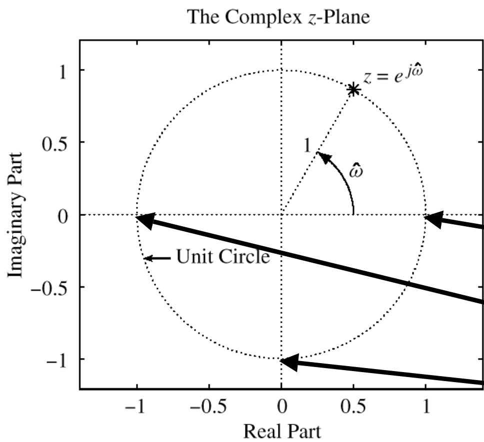


$$\frac{1-z^{-2}}{1+0.7225z^{-2}}$$



UNIT CIRCLE

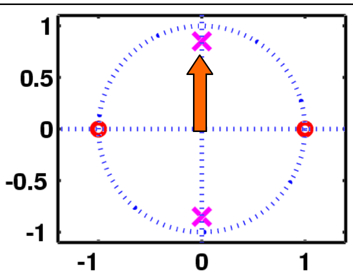
- **MAPPING BETWEEN** z and $\hat{\omega}$



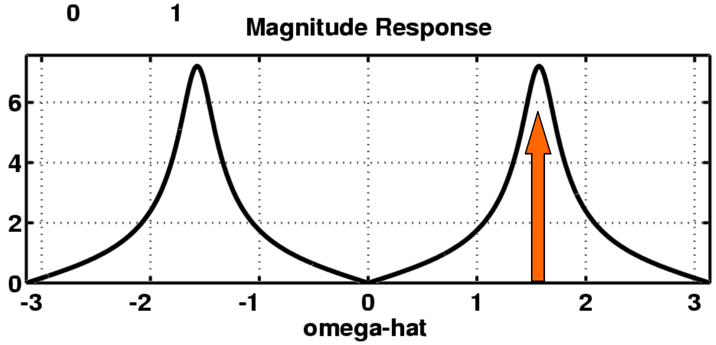
$$z = e^{j\hat{\omega}}$$

$z=1$	$\leftrightarrow$	$\hat{\omega}=0$
$z=-1$	$\leftrightarrow$	$\hat{\omega}=\pm\pi$
$z=\pm j$	$\leftrightarrow$	$\hat{\omega}=\pm\frac{1}{2}\pi$

FREQUENCY RESPONSE from POLE-ZERO PLOT



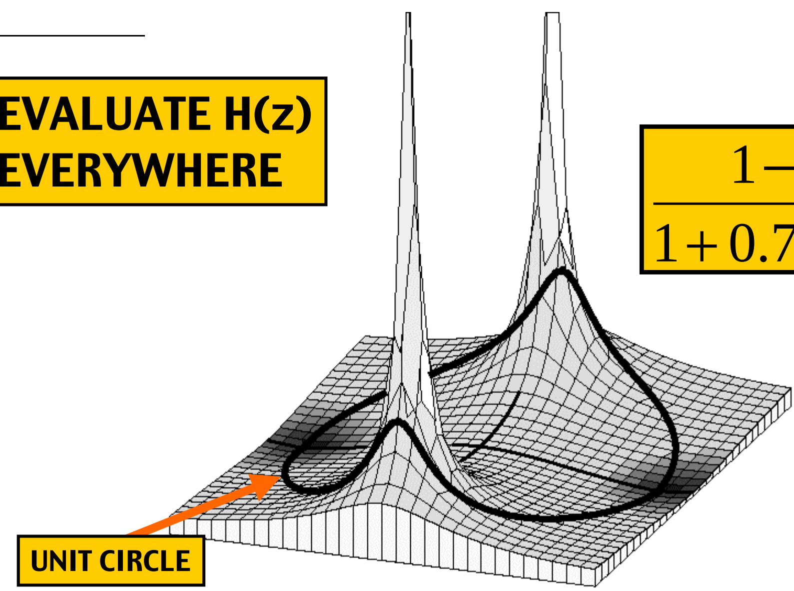
$$H(e^{j\hat{\omega}}) = \frac{1-e^{-j2\hat{\omega}}}{1+0.7225e^{-j2\hat{\omega}}}$$



3-D VIEW

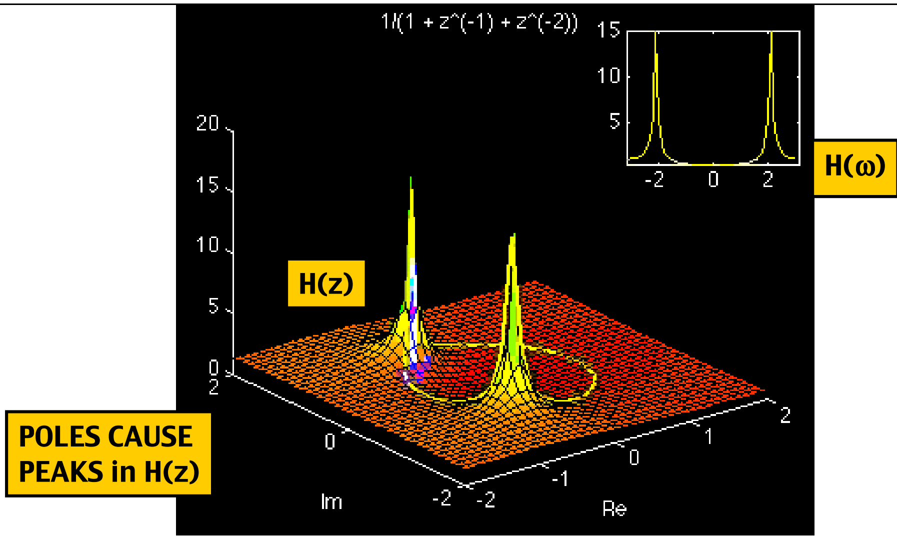
EVALUATE H(z) EVERYWHERE

$$\frac{1-z^{-2}}{1+0.7225z^{-2}}$$

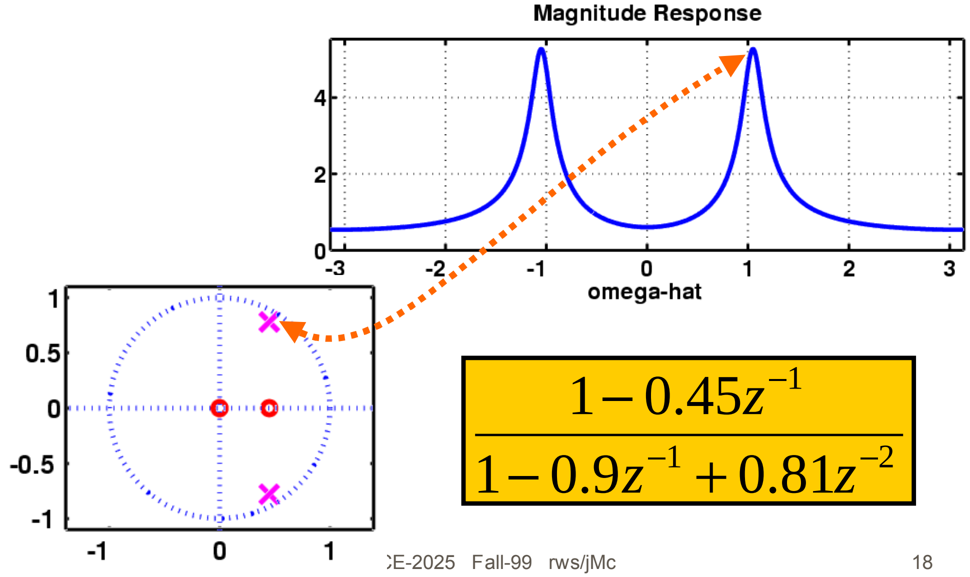


The poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.

FLYING THRU Z-PLANE

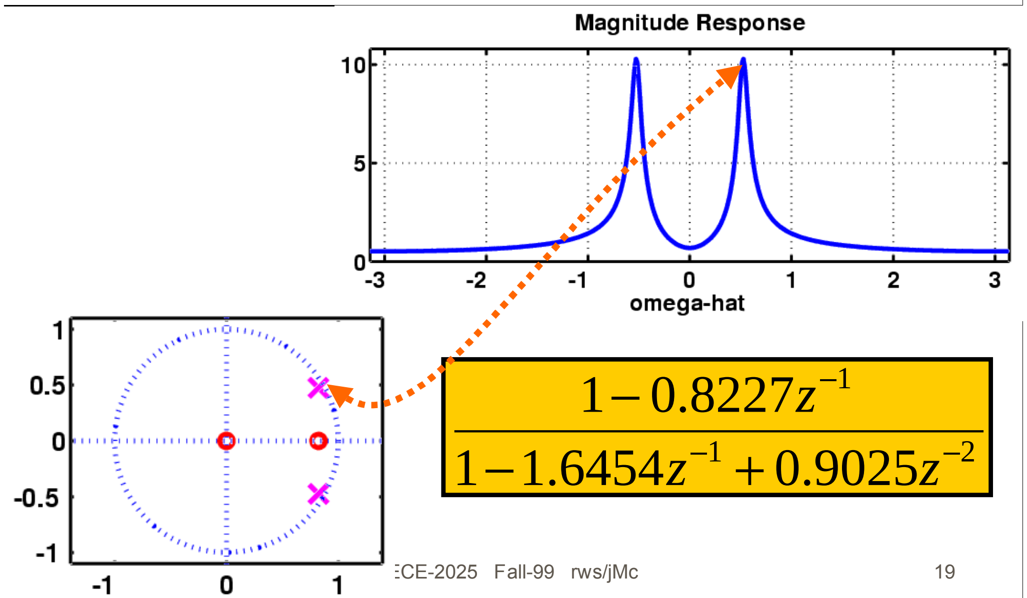


Complex POLE-ZERO PLOT



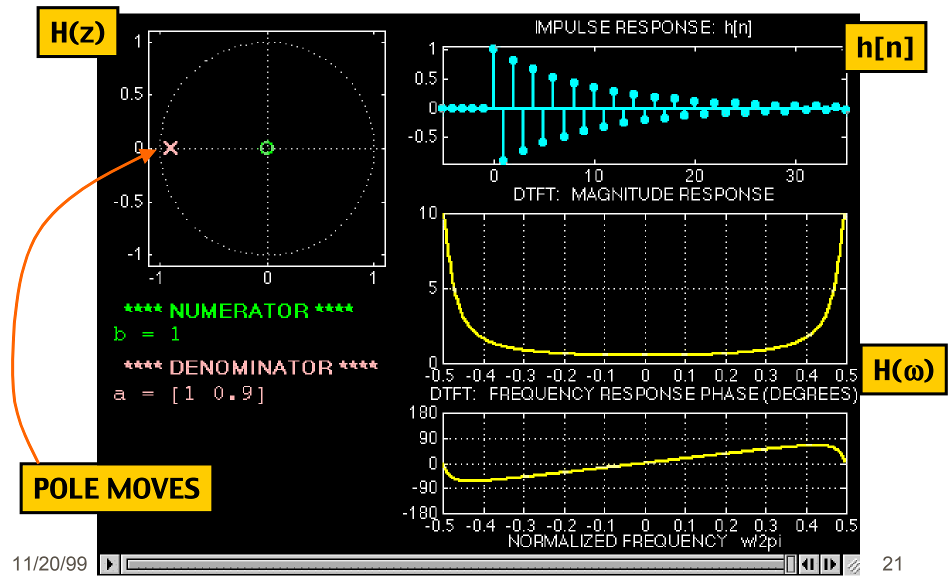
$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

Complex POLE-ZERO PLOT



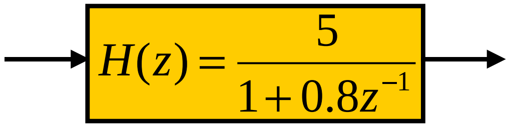
$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

3 DOMAINS MOVIE: IIR



THREE INPUTS

- Given:



- Find the output, y[n]
  - When

$x[n] = \cos(0.2\pi n)$

$x[n] = a^n u[n]$

$x[n] = \cos(0.2\pi n)u[n]$

SINUSOIDAL RESPONSE

- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if } x[n] = e^{j\hat{\omega}n}, \text{ then}$$

$$y[n] = \mathcal{H}(\hat{\omega})e^{j\hat{\omega}n}$$

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

SINUSOID ANSWER

- Given: $H(z) = \frac{5}{1+0.8z^{-1}}$
- The input: $x[n] = \cos(0.2\pi n)$
- Then y[n] $y[n] = M\cos(0.2\pi n + \psi)$

$$H(e^{j0.25\pi}) = \frac{5}{1+0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$

SINUSOID starts at n=0

$$x[n] = e^{j\hat{\omega}_0 n}u[n]$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

$H(z)$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

SPLIT Y(z) to INVERT

- Need **SUM** of Terms:

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

$$= \frac{b_0}{(1 - a_1 z^{-1})(1 - e^{j\hat{\omega}_0} z^{-1})}$$

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

INVERT Y(z) to y[n]

- Use the Z-Transform Table

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

TWO PARTS of y[n]

- **TRANSIENT**
  - Acts Like (pole)^n
  - Dies out ?
    - IF $|a_1|<1$

$$\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right) a_1^n u[n]$$

- **STEADY-STATE**
  - Depends on the input
  - e.g., Sinusoidal

$$\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

STEADY STATE HAPPENS

- When Transient dies out
- Limit as "n" approaches infinity
- Use Frequency Response to get Magnitude & Phase for sinusoid

$$y_{ss}[n] \rightarrow \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

NUMERICAL EXAMPLE

Example 8.12 If $b_0 = 5$, $a_1 = -0.8$, and $\hat{\omega}_0 = 2\pi/10$, the transient component is

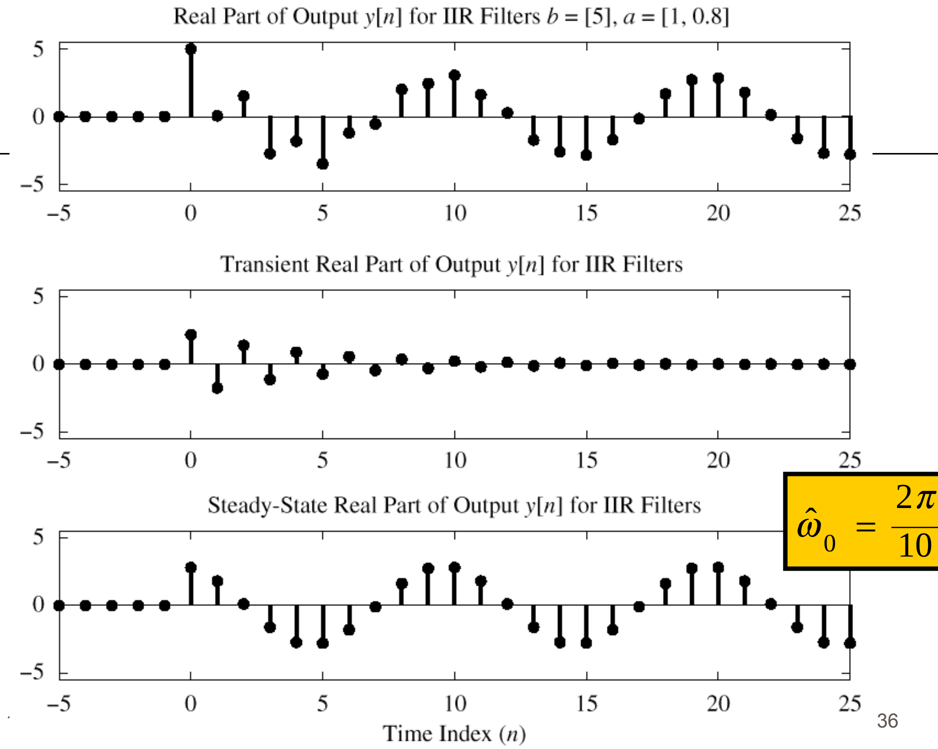
$$y_t[n] = \left(\frac{-4}{-0.8 - e^{j0.2\pi}}\right)(-0.8)^n u[n] = 2.3351 e^{-j0.3502}(-0.8)^n u[n]$$

$$= 2.1933(-0.8)^n u[n] - j0.8012(-0.8)^n u[n]$$

Similarly, the steady-state component is

$$y_{ss}[n] = \left(\frac{5}{1 + 0.8e^{-j0.2\pi}}\right) e^{j0.2\pi n} u[n] = 2.9188 e^{j0.2781} e^{j0.2\pi n} u[n]$$

$$= 2.9188 \cos\left(\tfrac{2\pi}{10} n + 0.2781\right) u[n] + j\, 2.9188 \sin\left(\tfrac{2\pi}{10} n + 0.2781\right) u[n]$$



STABILITY

- **When Does the TRANSIENT DIE OUT ?**

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$

STABILITY CONDITION

- **ALL POLES INSIDE the UNIT CIRCLE**
- **UNSTABLE EXAMPLE:**

POLE @ z=1.1

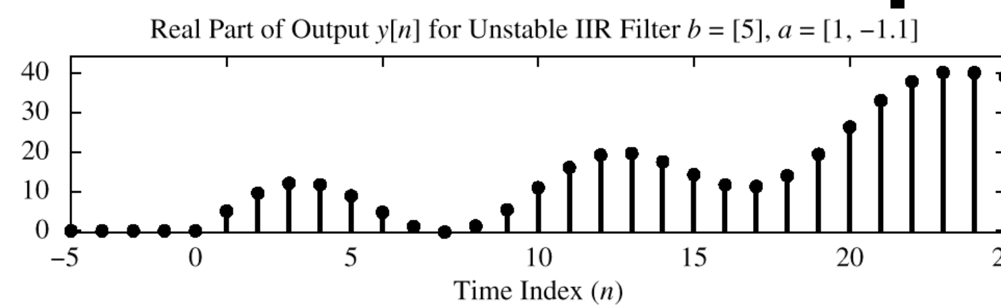


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.