EE-2025 Fall-99

Lecture 21

Introduction to the Fourier Transform

15-Nov-99

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 22-Nov
  - One page hand-written notes
  - Calculator
- Prob Set #12 is due Friday 19-Nov.
- Lab #12 on SYMBOLIC FOURIER SERIES
  - SYMBOLIC Toolbox
  - Then 1 more Lab: AM Communication

READING ASSIGNMENTS

- This Lecture:
  - Chapter 12, pp. 1200-1212
- Other Reading:
  - Recitation: Chapter 11
    - Ch. 12, pp. 1212-1222
  - Next Lecture: Chapter 12, pp. 1223-1234

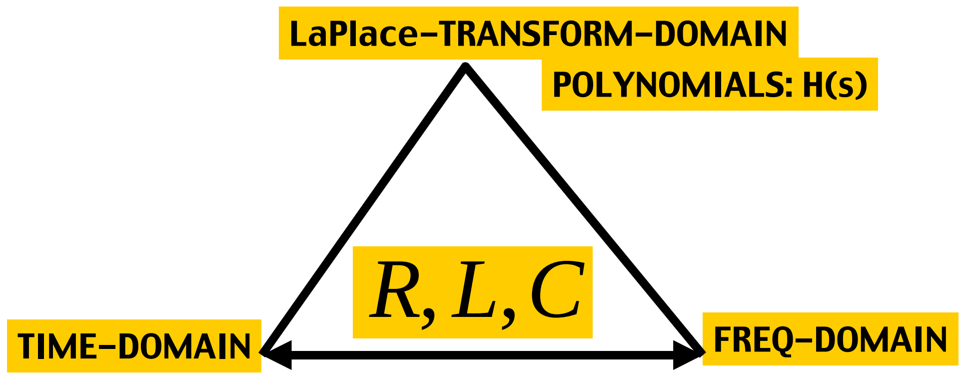
LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

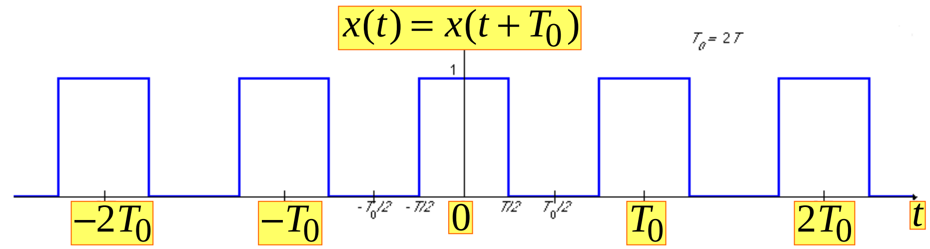
  - Relation to Fourier Series
- Examples of Fourier transform pairs

THREE DOMAINS: ANALOG



$y(t) = h(t) * x(t)$ ⟷ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Fourier Series: Periodic *x(t)*



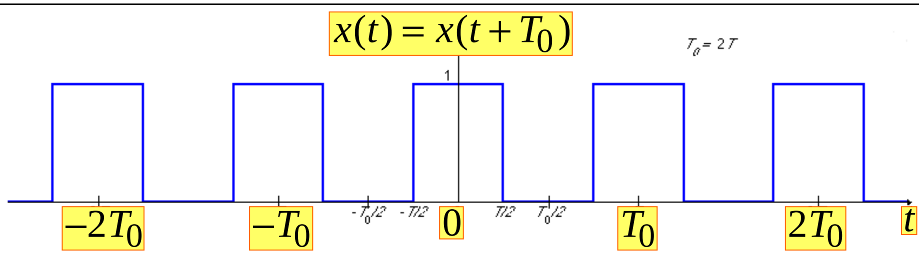
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$ ← **Fourier Synthesis**

Fundamental Freq.

$\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$ ← **Fourier Analysis**

Square Wave Signal

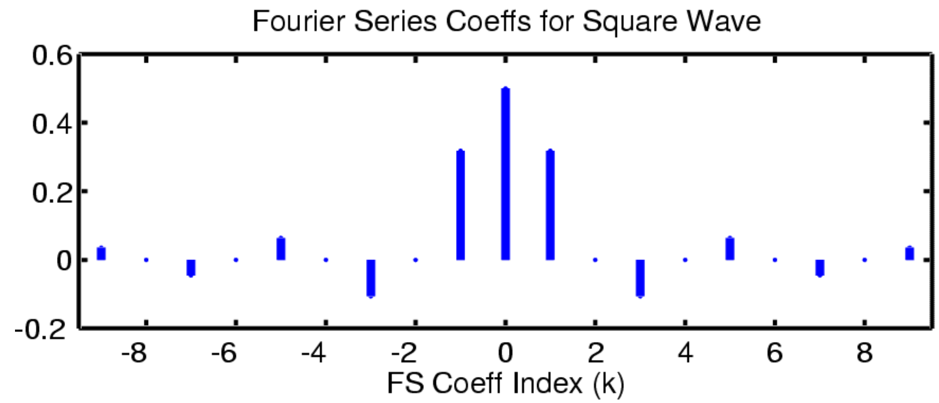


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Bigg|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

Spectrum from Fourier Series

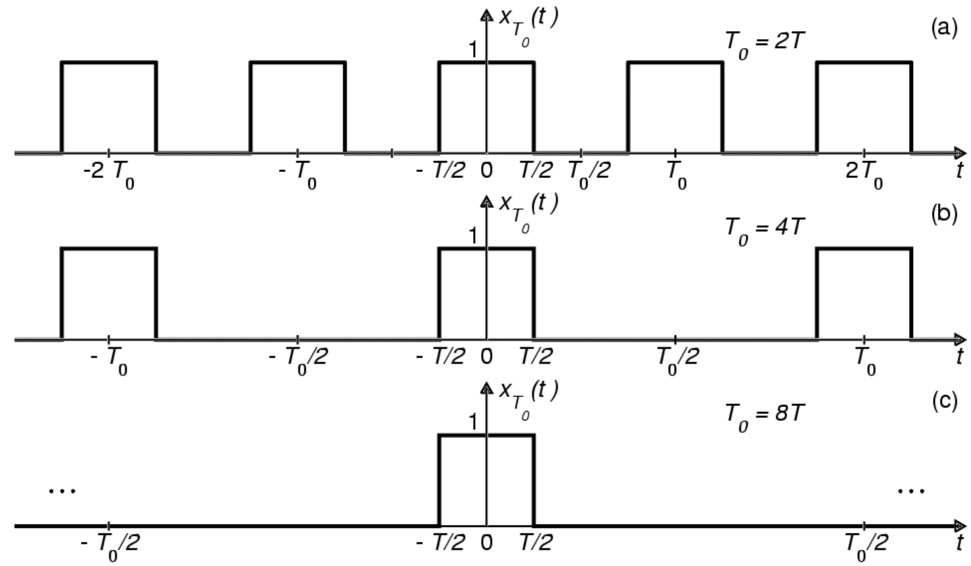
$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



What if *x(t)* is not periodic?

- **Sum of Sinusoids?**
  - **Non-harmonically related sinusoids**
  - **Would not be periodic, but would be non-zero for all *t*.**
- **Fourier transform**
  - **gives a "sum" (actually an integral) that involves ALL frequencies**
  - **can represent signals that are identically zero for negative *t*.**

Limiting Behavior of FS



The Fourier Transform

$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}(T_0 a_k)e^{j\omega_0 k t}\frac{2\pi}{T_0} \mapsto x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

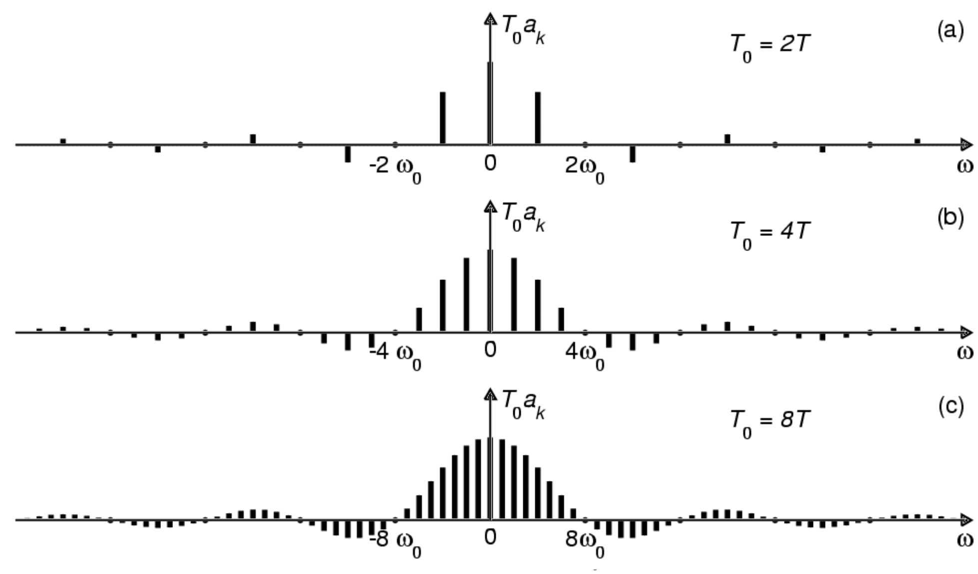
Fourier Synthesis

$$\lim_{T_0\to\infty}\frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0\to\infty}\frac{2\pi}{T_0}k = \omega$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-j\omega_0 k t}dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis

Limiting Behavior of Spectrum



Fourier Transform

- For non-periodic signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$ **Fourier Synthesis**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$ **Fourier Analysis**

Example 1: $x(t) = e^{-at}u(t)$

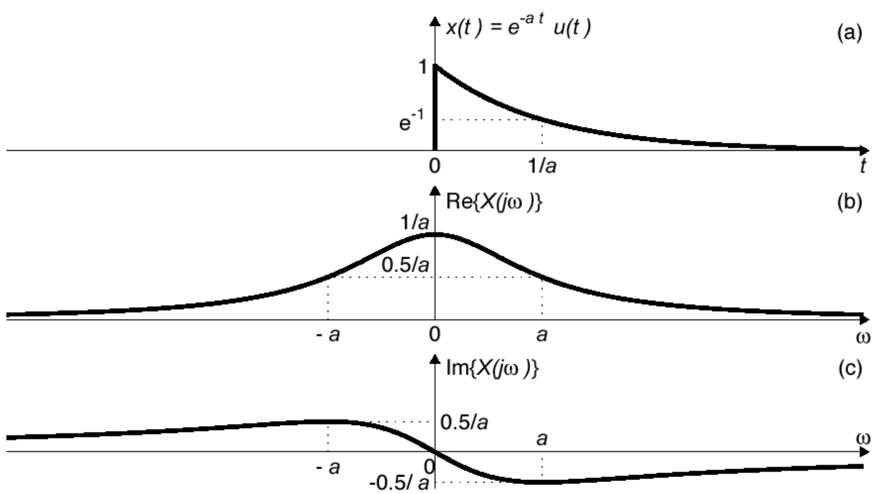
$$X(j\omega) = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$a > 0$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\Bigg|_0^{\infty} = \frac{1}{a+j\omega}$$

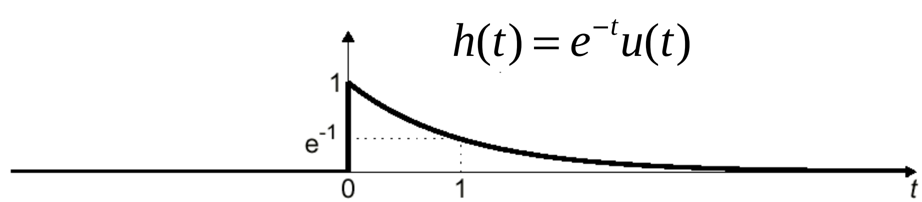
$$X(j\omega) = \frac{1}{a+j\omega}$$

$$x(t) = e^{-at}u(t) \quad\Leftrightarrow\quad X(j\omega) = \frac{1}{a+j\omega}$$



Frequency Response

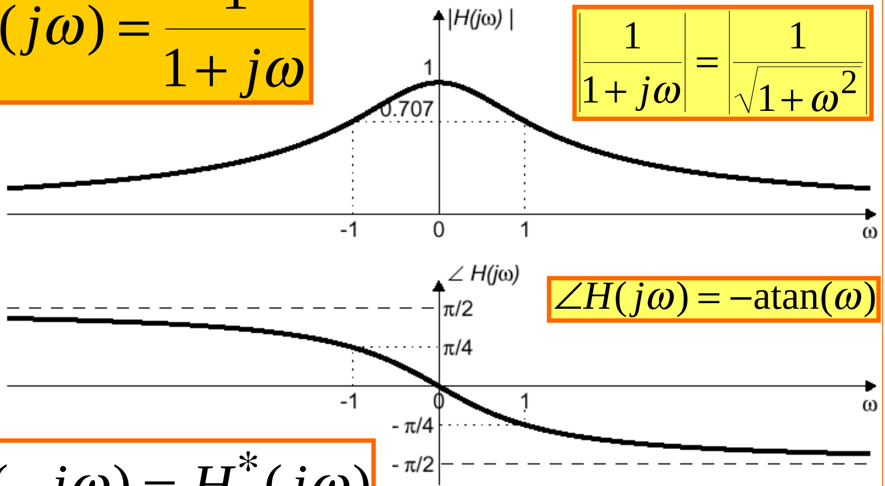
- Fourier Transform of h(t) is the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$



$$\left|\frac{1}{1+j\omega}\right| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle H(j\omega) = -\text{atan}(\omega)$$

$$H(-j\omega) = H^*(j\omega)$$

Example 2:

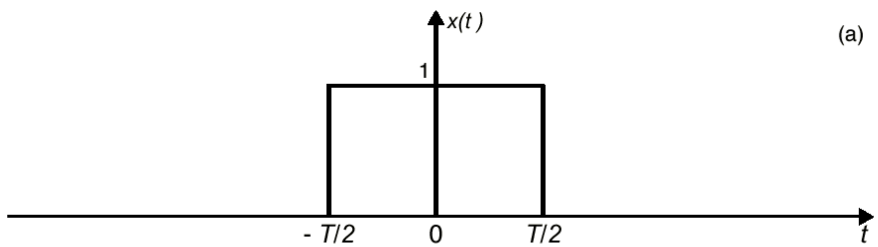
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

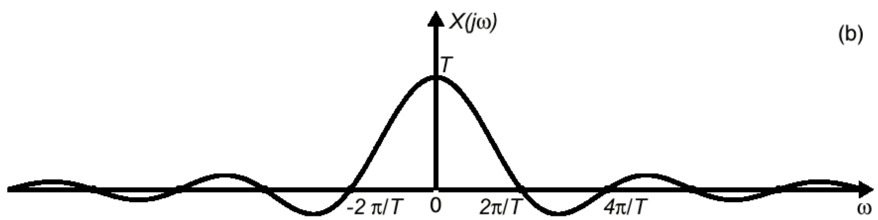
$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$





Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$