

EE-2025

Fall-99

Lecture 22

Fourier Transform Properties

19-Nov-99

Info: Web-CT, Lab, HW

- **Calendar: Quiz #3 is 22-Nov**
 - **REVIEW on SUNDAY at 7PM**
- **Prob Set #12 is due TODAY**
 - **Prob Set #13 posted soon, due 3-Dec**
- **NEXT WEEK:**
 - **CoC-309 closed over Thanksgiving**
 - **No Lab except Monday Lab sections, which have to catch up on 22-Nov.**
 - **Attend any convenient Recitation Section**

READING ASSIGNMENTS

- **This Lecture:**
 - **Chapter 12, pp. 1223–1239**
- **Other Reading:**
 - **Recitation: All of Chapter 12**
 - **Next Lecture: Chapter 13, pp. 1317–1332**

LECTURE OBJECTIVES

- **The Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
- **More examples of Fourier transform pairs**
- **Basic properties of Fourier transforms**
 - **Convolution property**
 - **Multiplication property**

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

Time - domain \Leftrightarrow Frequency - domain

$$x(t) \Leftrightarrow X(j\omega)$$

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Strategy for use of FT

- Develop a set of known Fourier transform pairs.
- Develop a set of “theorems” or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, **which ever leads to the simplest solution.**

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WHY use the Fourier transform?

- Manipulate the **“Frequency Spectrum”**
- Analog Communication Systems
 - AM: Amplitude Modulation; FM
 - What are the **“Building Blocks”**?
 - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
 - Modulators, or Multipliers: $x(t)p(t)$

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Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

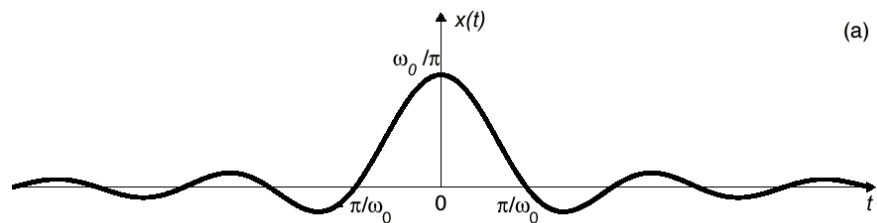
$$x(t) = \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

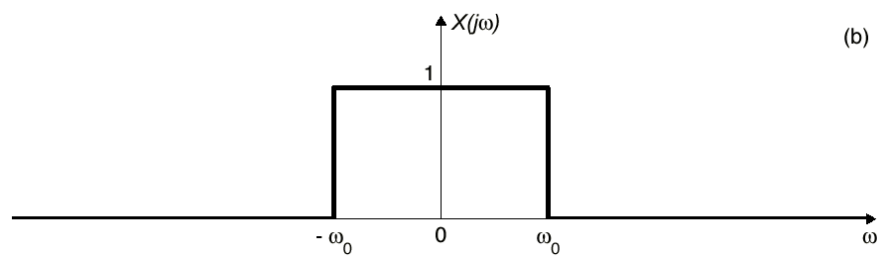
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$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



(a)



(b)

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Example 4: $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

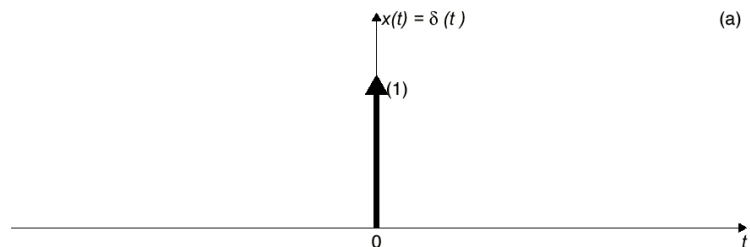
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

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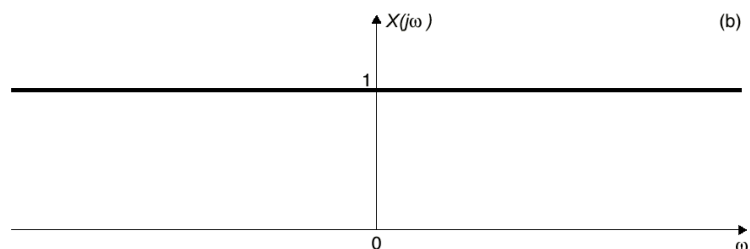
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$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



(a)



(b)

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Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

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Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

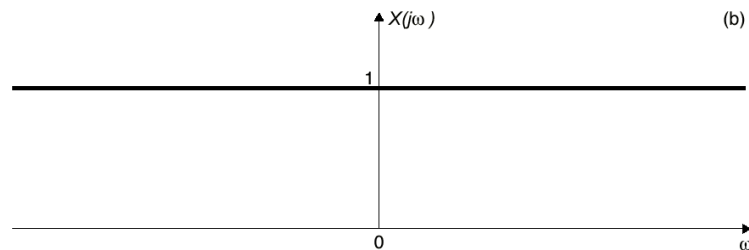
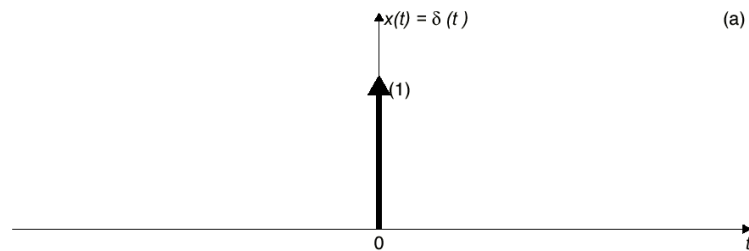
$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

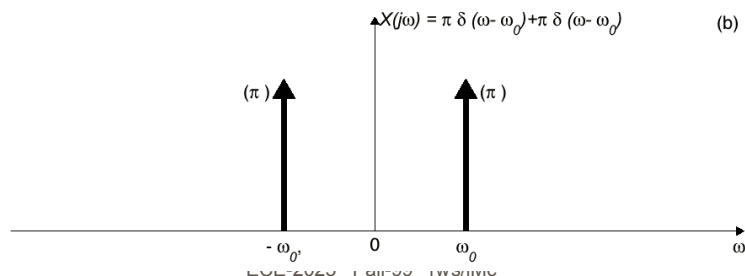
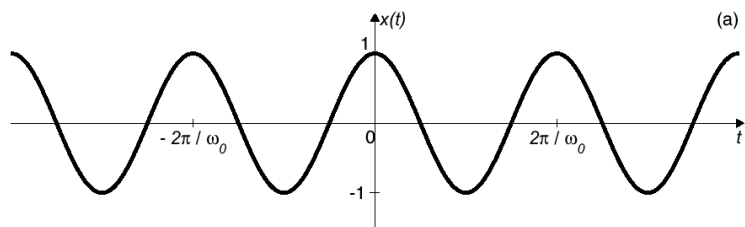


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$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



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Fourier Transform of General Periodic Signal

■ If $x(t)$ is periodic with period T_0 ,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

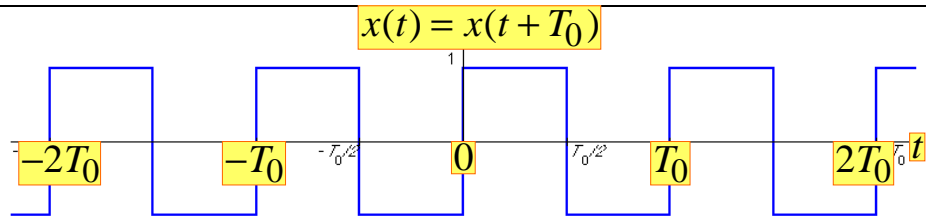
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

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Square Wave Signal



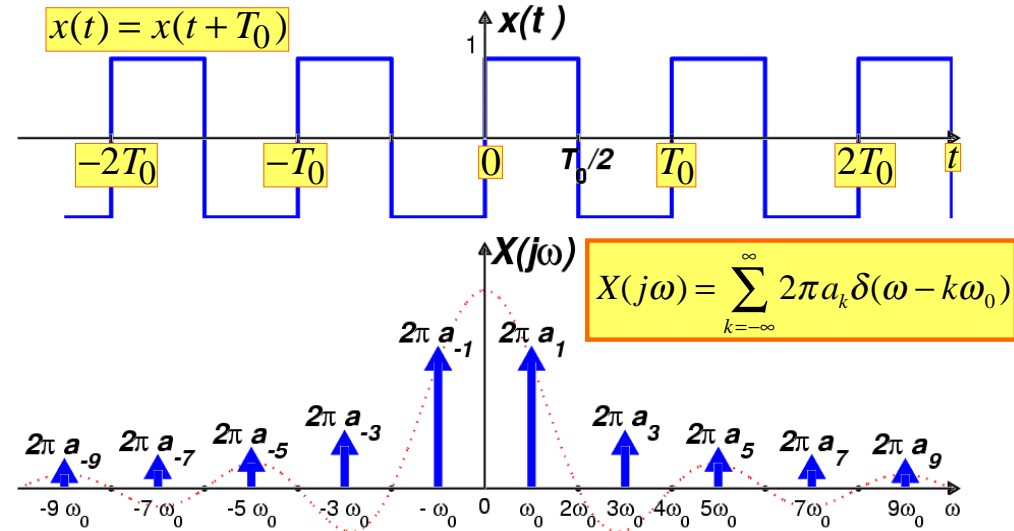
$$a_k = \frac{1}{T_0} \int_0^{T_0/2} (1) e^{-j\omega_0 kt} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 kt} dt$$

$$a_k = \frac{e^{-j\omega_0 k T_0/2}}{-j\omega_0 k T_0} \Big|_0^{T_0/2} - \frac{e^{-j\omega_0 k T_0}}{-j\omega_0 k T_0} \Big|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

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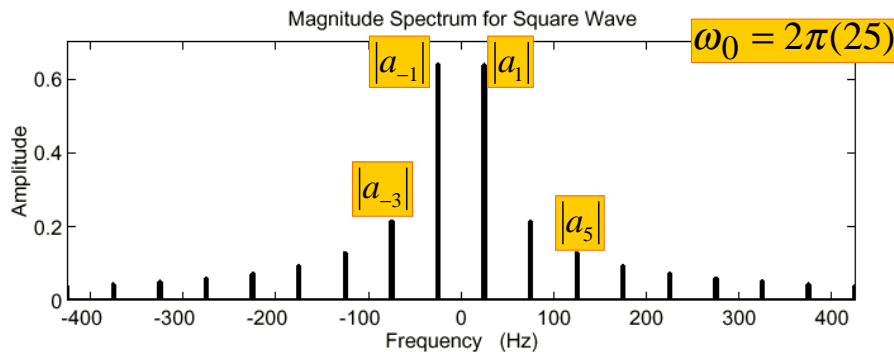
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Square Wave SPECTRUM



Spectrum from Fourier Series

$$a_k = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} \frac{2}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = 0, \pm 2, \pm 4, \dots \end{cases}$$



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Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

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Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

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Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

$$\begin{aligned} \int_{-\infty}^{\infty} x(t - t_d) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_d)} d\tau \\ &= e^{-j\omega t_d} X(j\omega) \end{aligned}$$

For example, $e^{-a(t-5)} u(t-5) \Leftrightarrow \frac{e^{-j\omega 5}}{a + j\omega}$

1

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-j\omega_0 t} x(t) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt \\ &= X(j(\omega - \omega_0)) \end{aligned}$$

$$y(t) = \frac{\sin 7t}{\pi} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & \text{elsewhere} \end{cases}$$

Differentiation Property

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} (e^{-at} u(t)) &= -ae^{-at} u(t) + e^{-at} \delta(t) \\ &= \delta(t) - ae^{-at} u(t) \end{aligned}$$

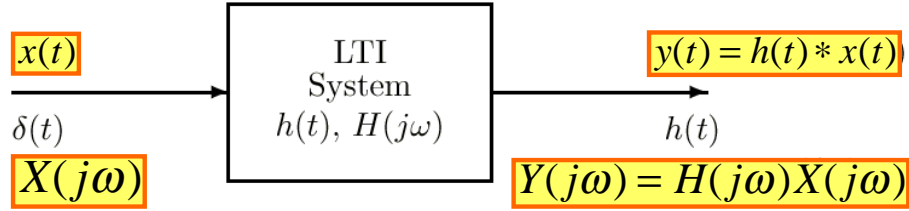
$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

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Convolution Property



Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

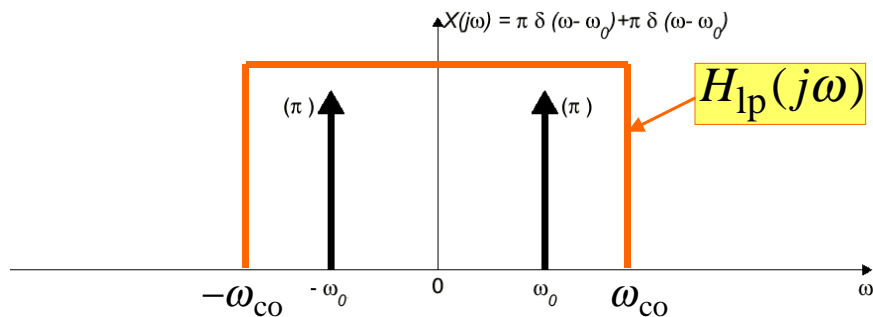
corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Cosine Input to LTI System

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] \\ &= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0) \end{aligned}$$

$$\begin{aligned} y(t) &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

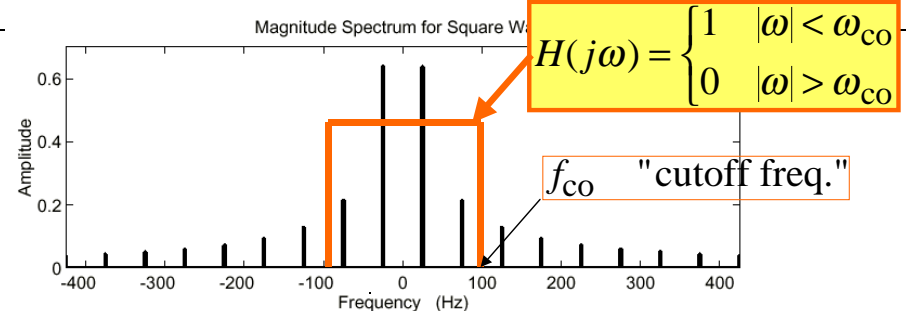
Ideal Lowpass Filter



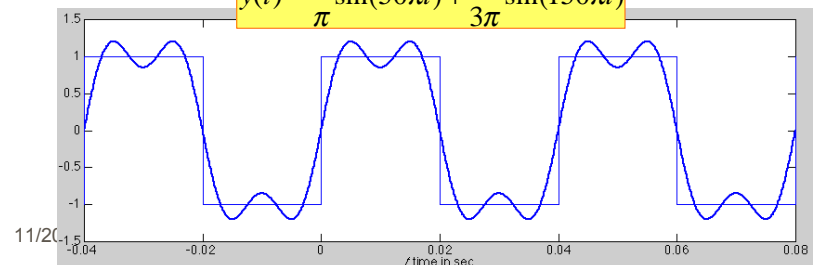
$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$

