EE-2025 Fall-99

Lecture 25

Discrete-Time Filtering of Continuous-Time Signals

6-Dec-99

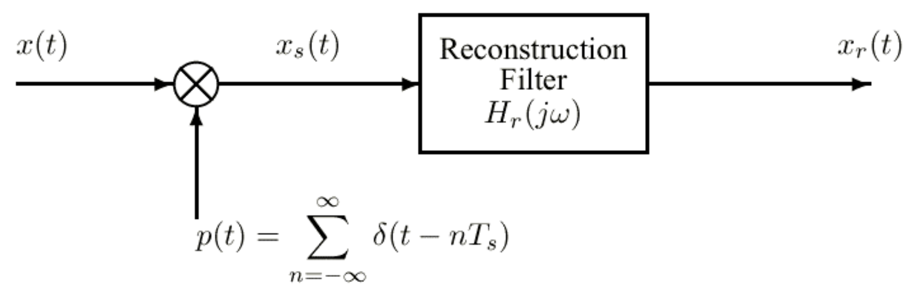
Info: Web-CT, Lab, HW

- Calendar: Final Exams
  - 11am section–Period 9, Weds, 12/15
  - 12pm section–Period 13, Fri, 12/17
  - NO switching allowed
- Reading Assignment:
  - You should have read Chapters 2–8 of DSP First and all chapters of the notes.
- Prob Set #14 – not handed in;
  - However, it will be discussed in recitations and covered on FINAL EXAM

LECTURE OBJECTIVES

- Discrete–time filtering of continuous–time signals
  - Basic configuration
    - CT Input –> A/D –> DT System –> D/A –> CT Output
  - Equivalent CT frequency response
- Review of FT properties
  - Convolution <––> multiplication
  - Frequency shifting
  - Review of sampling

Impulse Train Sampling

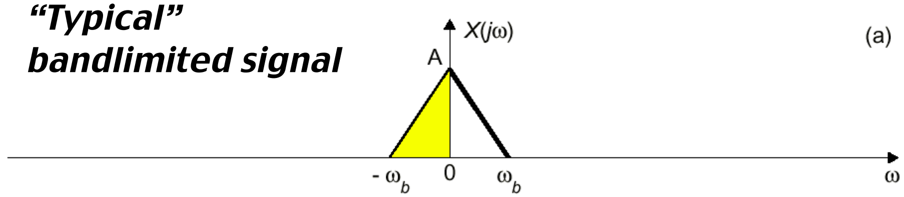


$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

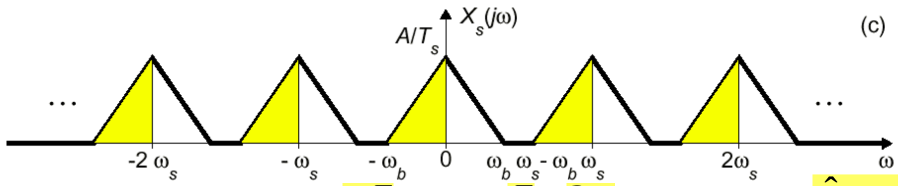
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

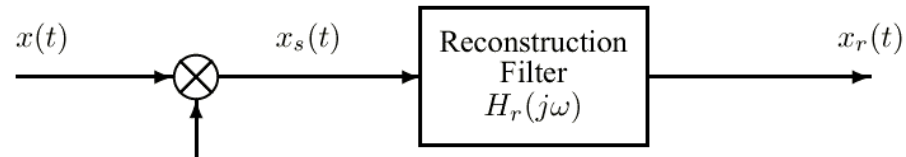


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



$-\pi$ $\quad \pi \quad 2\pi \qquad \hat{\omega} = \omega T_s$

Reconstruction of $x(t)$

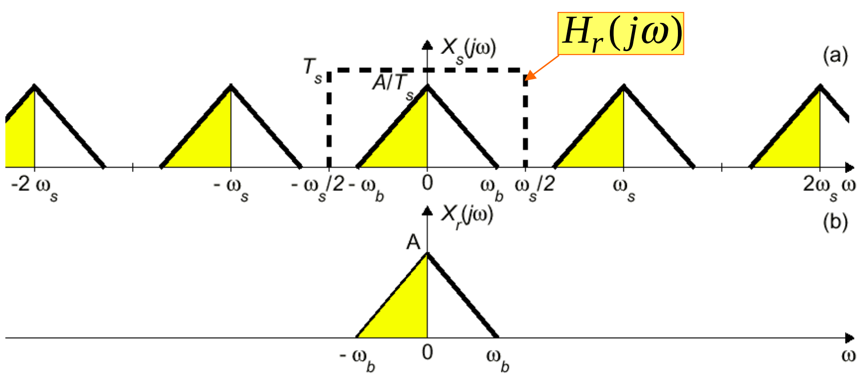


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

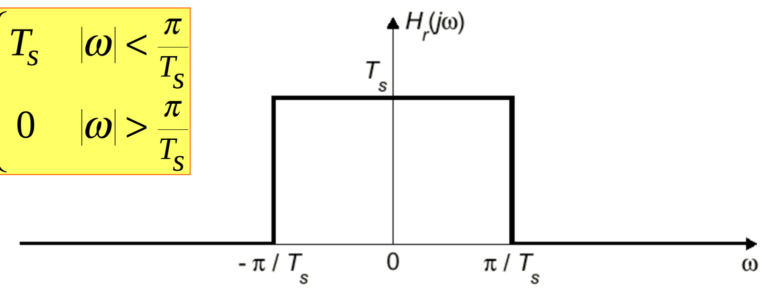
Reconstruction in the Frequency-Domain



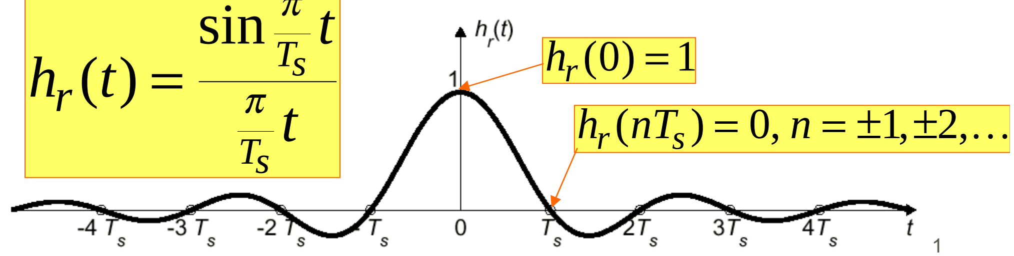
- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)\, X_s(j\omega)$.

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



Shannon Sampling Theorem

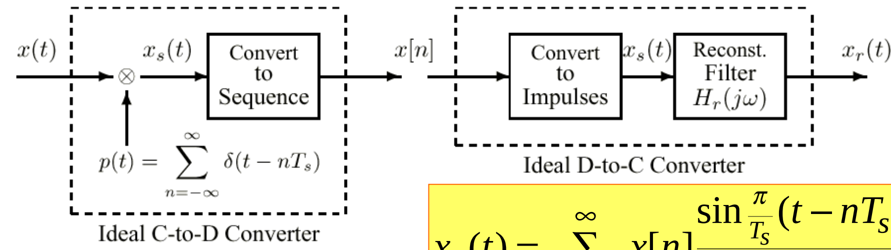
- **PERFECT RECONSTRUCTION of BANDLIMITED SIGNALS**

A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

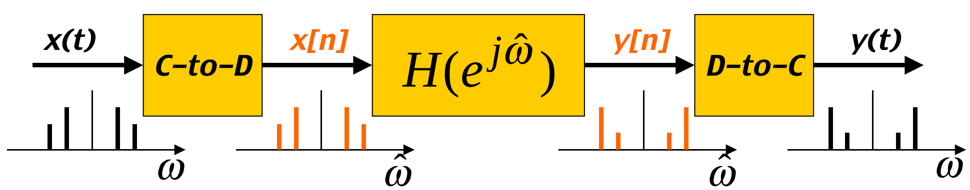
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

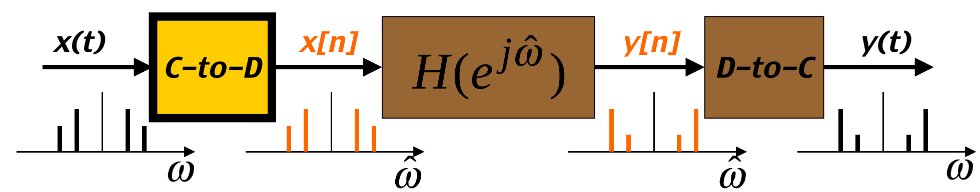
$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

DIGITAL "FILTERING"



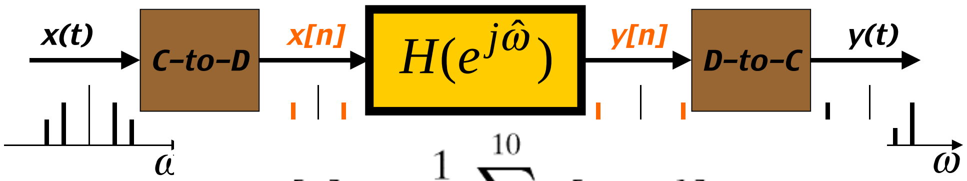
- ω **SPECTRUM of $x(t)$ (FOURIER TRANSFORM)**
- $\hat{\omega}$ **SPECTRUM of $x[n]$**
  - **Is ALIASING a PROBLEM ?**
- **SPECTRUM $y[n]$ (FIR Gain or Nulls)**
- ω **Then, OUTPUT $y(t)$ = SUM of SINUSOIDS**

FREQUENCY SCALING



- **TIME SAMPLING:** $t = nT_s$
  - **IF NO ALIASING:**
  - **FREQUENCY SCALING** $\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$

11-pt AVERAGER Example



$$y[n] = \frac{1}{11}\sum_{k=0}^{10} x[n-k]$$

250 Hz

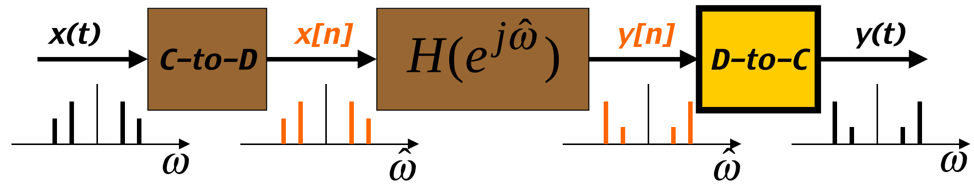
25 Hz

$$H(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)}e^{-j\hat{\omega}5}$$

?

$$x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$$

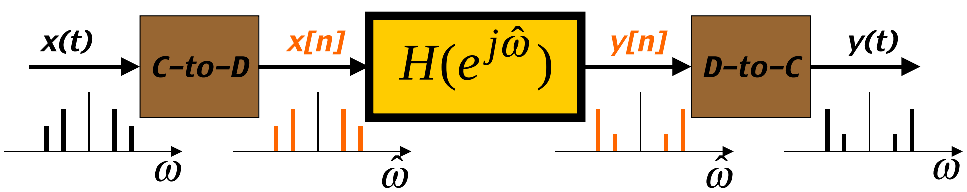
D-A FREQUENCY SCALING



- **TIME SAMPLING** $t = nT_s \Rightarrow n \leftarrow t f_s$
- **RECONSTRUCT up to $0.5f_s$**
  - **FREQUENCY SCALING**

$$\omega = \hat{\omega} f_s$$

TRACK the FREQUENCIES

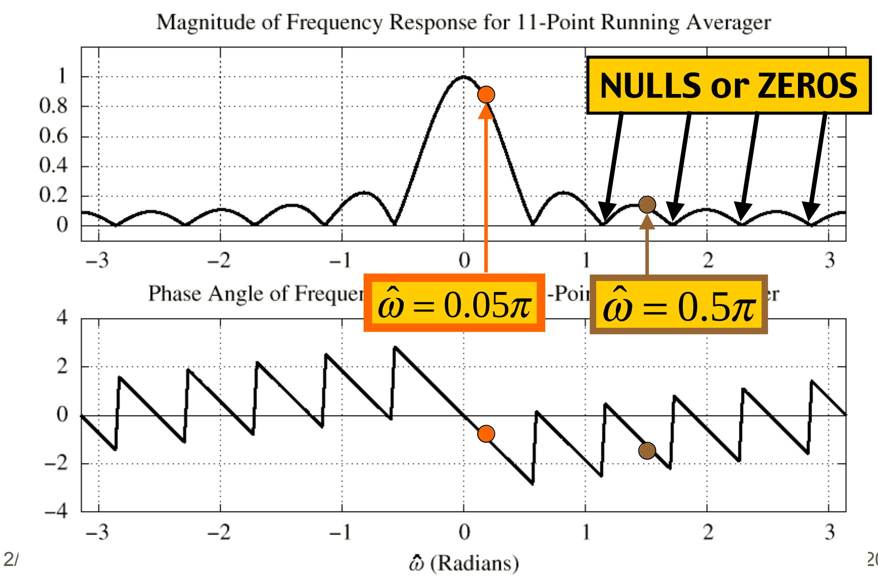


250 Hz	0.5π	$H(e^{j0.5\pi})$	0.5π	250 Hz
25 Hz	$.05\pi$	$H(e^{j0.05\pi})$	$.05\pi$	25 Hz

F_s = 1000 Hz

NO new freqs

11-pt AVERAGER



EVALUATE Freq. Response

$$H(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)} e^{-j\hat{\omega}5}$$

At $\hat{\omega} = 0.5\pi$

$$H(e^{j0.5\pi}) = \frac{\sin((0.5\pi)11/2)}{11\sin(0.5\pi/2)} e^{-j(0.5\pi)5}$$

$$= \frac{\sin(2.75\pi)}{11\sin(0.25\pi)} e^{-j2.5\pi}$$

$$= 0.0909 e^{-j0.5\pi}$$

EVALUATE Freq. Response

$$x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$$

evaluating at 25 and 250 Hz.

$f_s = 1000$

$$H(e^{j2\pi(25)/1000}) = \frac{\sin(\pi(25)(11)/1000)}{11\sin(\pi(25)/1000)} e^{-j2\pi(25)(5)/1000}$$

$$= 0.8811 e^{-j\pi/4}$$

$$H(e^{j2\pi(250)/1000}) = \frac{\sin(\pi(250)(11)/1000)}{11\sin(\pi(250)/1000)} e^{-j2\pi(250)(5)/1000}$$

$$= 0.0909 e^{-j\pi/2}$$

MAG SCALE

PHASE CHANGE

$$y(t) = 0.8811\cos(2\pi(25)t - \pi/4) + 0.0909\sin(2\pi(250)t - \pi/2)$$

General Frequency-Domain Analysis of C-to-D Conversion

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

$$x[n] = x(nT_s) \qquad \hat{\omega} = \omega T_s$$

Discrete-Time Fourier Transform and *z*-Transform

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}$$

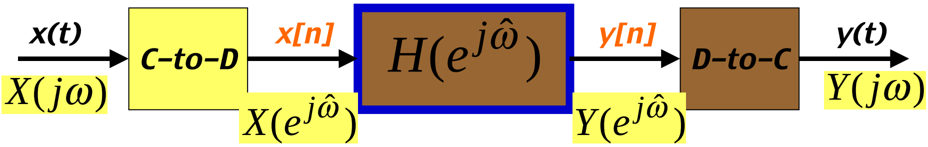
$\hat{\omega} = \omega T_s$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$ ***z*–Transform**

$$X(z)\big|_{z=e^{j\omega T_s}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n} = X(e^{j\omega T_s})$$ ***DTFT***

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) = X(e^{j\omega T_s})$$

C-to-D Converter

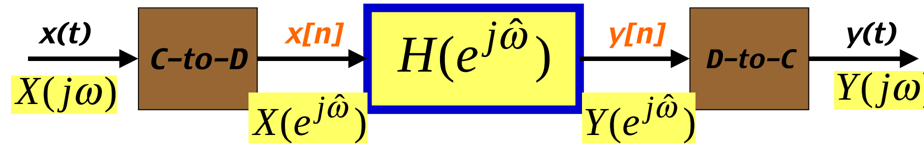


$$x[n] = x(nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X(e^{j\omega T_s}) = X(z)\Big|_{z=e^{j\omega T_s}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n} = X_s(j\omega)$$

LTI DT System



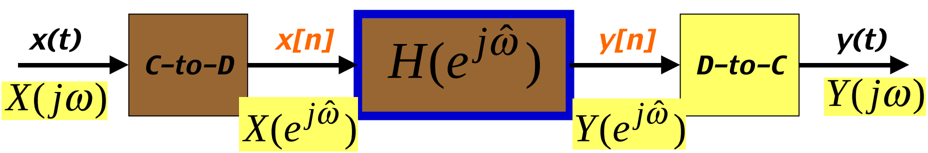
$$Y(z) = H(z)X(z)$$

$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

$$Y(e^{j\omega T_s}) = H(e^{j\omega T_s})X(e^{j\omega T_s})$$

D-to-C Converter

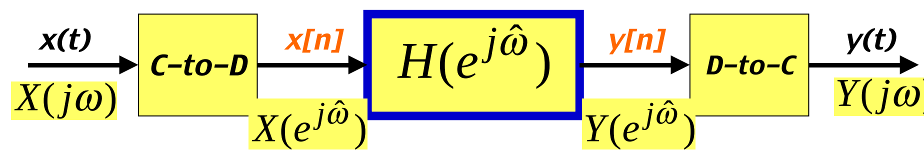


$$y(t) = \sum_{n=-\infty}^{\infty} y[n]h_r(t - nT_s)$$

$$Y(j\omega) = \sum_{n=-\infty}^{\infty} y[n]H_r(j\omega)e^{-j\omega T_s n} = H_r(j\omega)Y(e^{j\omega T_s})$$

$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s}) = H_r(j\omega)H(e^{j\omega T_s})X(e^{j\omega T_s})$$

Putting it All Together



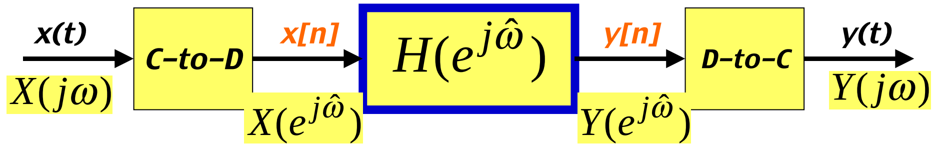
$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s}) = H_r(j\omega)H(e^{j\omega T_s})\underline{X(e^{j\omega T_s})}$$

$$Y(j\omega) = H_r(j\omega)H(e^{j\omega T_s})\underline{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H(e^{j\omega T_s})X(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

DT Filtering of CT Signals

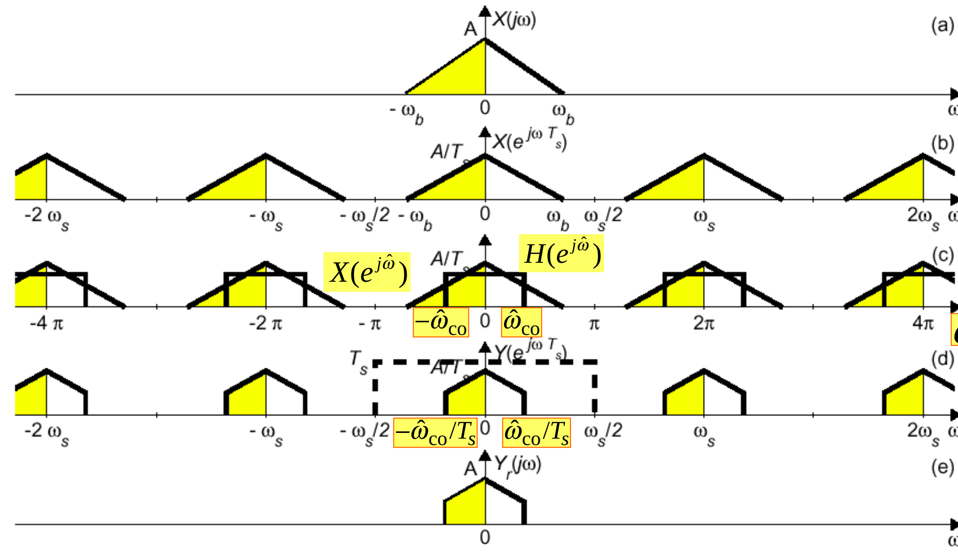


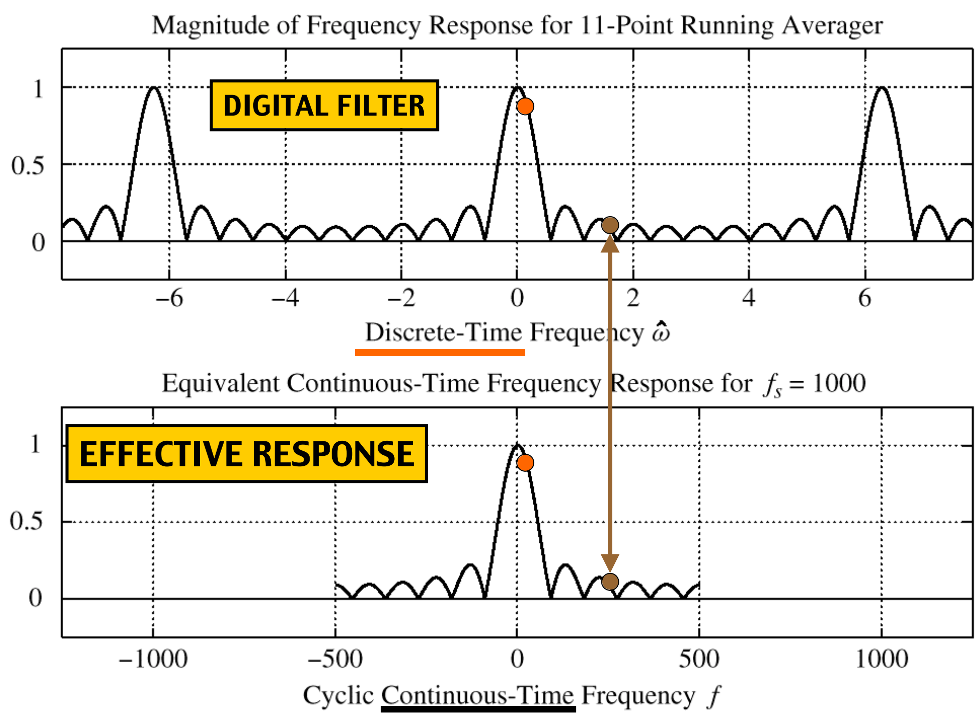
If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

Illustration of DT Filtering of a CT Signal





EFFECTIVE Freq. Response

- Assume NO Aliasing, then
  - ANALOG FREQ <---> DIGITAL FREQ

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DIGITAL FILTER

  - So, we can plot:
  - Scaled Freq. Axis

$$H(e^{j\omega T_s}) \text{ vs. } \omega$$

ANALOG FREQUENCY