GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

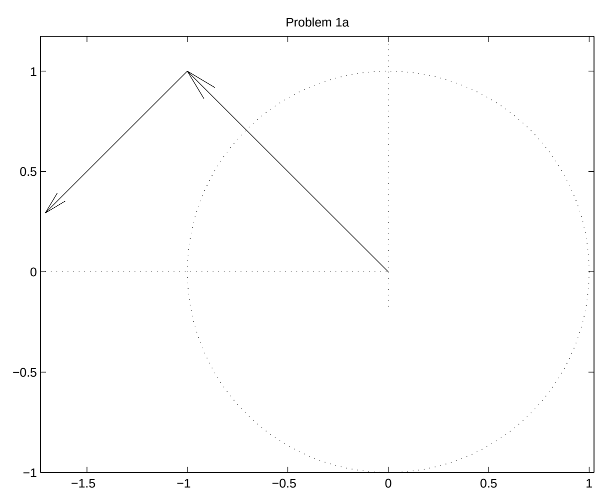
ECE 2025 Fall 1999
Problem Set #2

Assigned: 3 September 1999
Due Date: 10 September 1999 (FRIDAY)

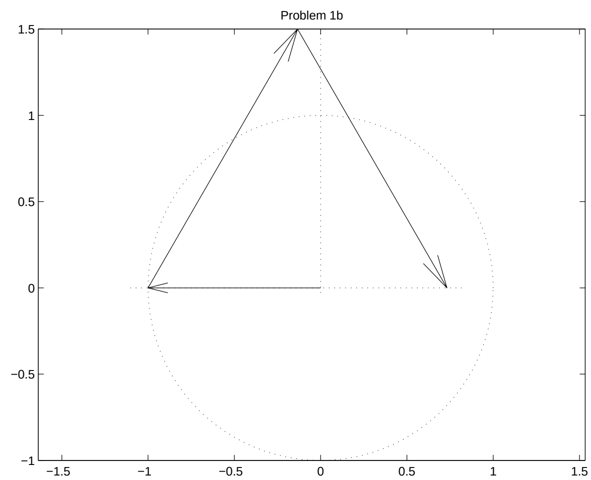
PROBLEM 2.1:

Simplify the following and give the answer as a single sinusoid. Draw the vector diagram of the complex amplitudes (phasors) to show how you obtained the answer.

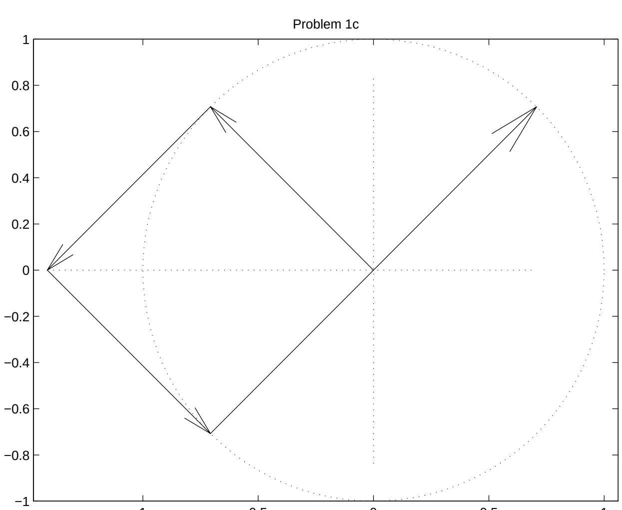
$$
\begin{aligned}
x_a(t) &= \sqrt{2}\cos(2\pi t + 3\pi/4) - \cos(2\pi t + \pi/4) \\
z_a(t) &= \sqrt{2}e^{j3\pi/4}e^{j2\pi t} + (-1)e^{j\pi/4}e^{j2\pi t} \\
&= [\sqrt{2}e^{j3\pi/4} + (-1)e^{j\pi/4}]e^{j2\pi t} \\
&= [(-1+j) + (-\sqrt{2}/2 - j\sqrt{2}/2)]e^{j2\pi t} \\
&= [\frac{-2-\sqrt{2}}{2} + j\frac{2-\sqrt{2}}{2}]e^{j2\pi t} \\
&\approx (1.73 \cdot e^{j0.946\pi})e^{j2\pi t} \\
x_a(t) &\approx 1.73\cos(2\pi t + 0.946\pi)
\end{aligned}
$$



$$
\begin{aligned}
x_b(t) &= \cos(11t + 17\pi) + \sqrt{3}\cos(11t + \pi/3) + \sqrt{3}\cos(11t - \pi/3) \\
z_b(t) &= (e^{j17\pi} + \sqrt{3}e^{j\pi/3} + \sqrt{3}e^{-j\pi/3})e^{j11t} \\
&= [(-1) + (\sqrt{3}/2 + j3/2) + (\sqrt{3}/2 - j3/2)]e^{j11t} \\
&= [\sqrt{3} - 1]e^{j11t} \\
&\approx 0.732e^{j11t} \\
x_b(t) &\approx 0.732\cos(11t)
\end{aligned}
$$



$$
\begin{aligned}
x_c(t) &= \cos(\pi t + 3\pi/4) + \cos(\pi t + 5\pi/4) + \cos(\pi t - \pi/4) + 2\cos(\pi t + \pi/4) \\
z_c(t) &= (e^{j3\pi/4} + e^{j5\pi/4} + e^{-j\pi/4} + 2e^{j\pi/4})e^{j\pi t} \\
&= [\frac{\sqrt{2}}{2}(-1 + j) + \frac{\sqrt{2}}{2}(-1 - j) + \frac{\sqrt{2}}{2}(1 - j) + \sqrt{2}(1 + j)]e^{j\pi t} \\
&= [\frac{\sqrt{2}}{2}(1 + j)]e^{j\pi t} \\
&= [e^{j\pi/4}]e^{j\pi t} \\
x_c(t) &= \cos(\pi t + \pi/4)
\end{aligned}
$$



PROBLEM 2.2:

Define $x(t)$ as

$$x(t) = 20\cos(200\pi t + \pi/2) + A\cos(200\pi t + \phi) \quad (1)$$

How should A and ϕ be chosen so that

$$x(t) = B\cos(200\pi t), \quad (2)$$

where B is a positive real number? What is the value of B for your choice of A and ϕ?

In order to solve the problem, we set $x(t)$ in (1) equal to $x(t)$ in (2) as follows:

$$20\cos(200\pi t + \pi/2) + A\cos(200\pi t + \phi) = B\cos(200\pi t)$$

From this equation, we can derive the following using the Phasor Addition Rule:

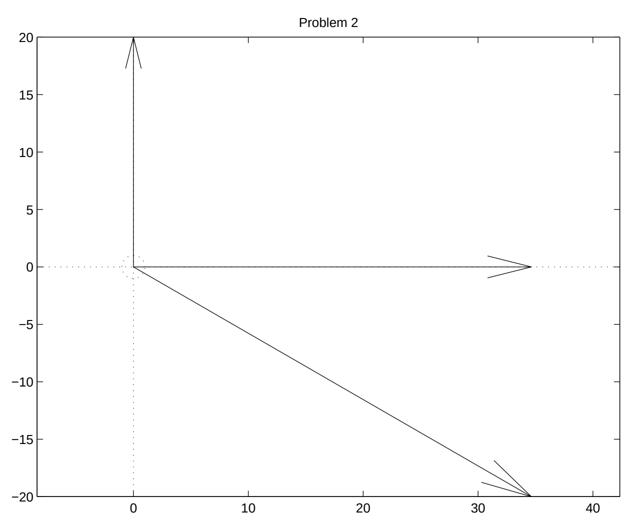
$$\begin{aligned} 20e^{j\pi/2} + Ae^{j\phi} &= B \\ j20 + [A\cos(\phi) + jA\sin(\phi)] &= B \end{aligned}$$

By separating the real and imaginary parts of this equation, we get

$$A\cos(\phi) = B \quad (3)$$

$$A\sin(\phi) = -20 \quad (4)$$

Thus, in order to guarantee that B is a positive number, we must choose A and ϕ as follows: The magnitude of A must be greater than 20 in order to satisfy (4). If A is chosen such that $A > 20$, ϕ must be chosen so that $\sin(\phi) = -20/A < 0$, and $\cos(\phi) = B/A > 0$, resulting in $-\pi/2 < \phi < 0$. If A is chosen such that $A < -20$, ϕ must be chosen so that $\sin(\phi) = -20/A > 0$, and $\cos(\phi) = B/A < 0$, resulting in $\pi/2 < \phi < \pi$. In either case, the resulting vector, $Ae^{j\phi}$, is **in the fourth quadrant** as shown in the figure below, which uses the values $A = 40$, $B = 20\sqrt{3}$, and $\phi = -\pi/6$.

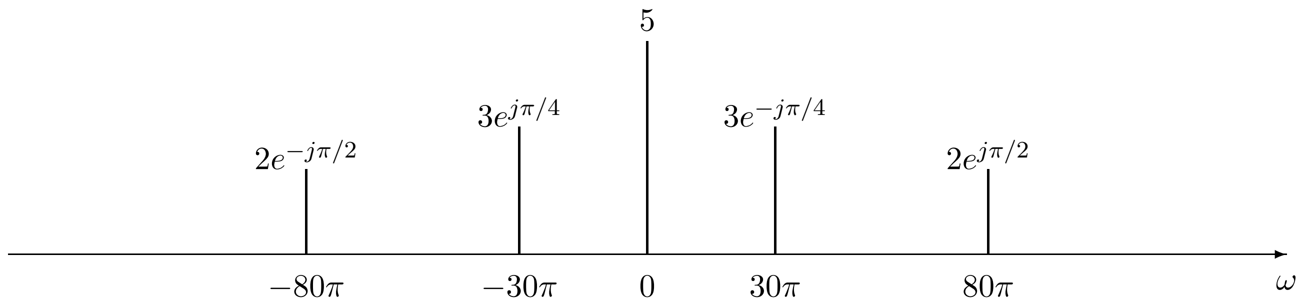


Additionally, if we are given B, we can calculate A (positive) and ϕ using the following equations derived by combining and rearranging (3) and (4):

$$\begin{aligned} A &= \sqrt{B^2 + 400} \\ \phi &= \tan^{-1}(-20/B) \end{aligned}$$

PROBLEM 2.3:

A real signal $x(t)$ has the following two-sided spectrum:



(a) Write an equation for $x(t)$ as a sum of cosines.
We take the magnitudes and phases from the positive frequency terms:

$$x(t) = 5 + 6\cos(30\pi t - \pi/4) + 4\cos(80\pi t + \pi/2)$$

(b) Explain why the "negative" frequencies are necessary to obtain a real signal.
We are deriving our sum of sinusoids from complex exponential signals of the form

$$Ae^{j(2\pi\omega t+\phi)} = A\cos(2\pi\omega t + \phi) + jA\sin(2\pi\omega t + \phi)$$

where the right side of the equation is derived from Euler's formula. However, we would like a sinusoidal signal of the form $A\cos(2\pi\omega t + \phi)$, effectively eliminating the imaginary part of the previous equation. This elimination can be achieved using the Inverse Euler Formula, which states

$$A\cos(2\pi\omega t + \phi) = \frac{A}{2}e^{j(2\pi\omega t+\phi)} + \frac{A}{2}e^{-j(2\pi\omega t+\phi)}$$

This equation can be rearranged slightly to give

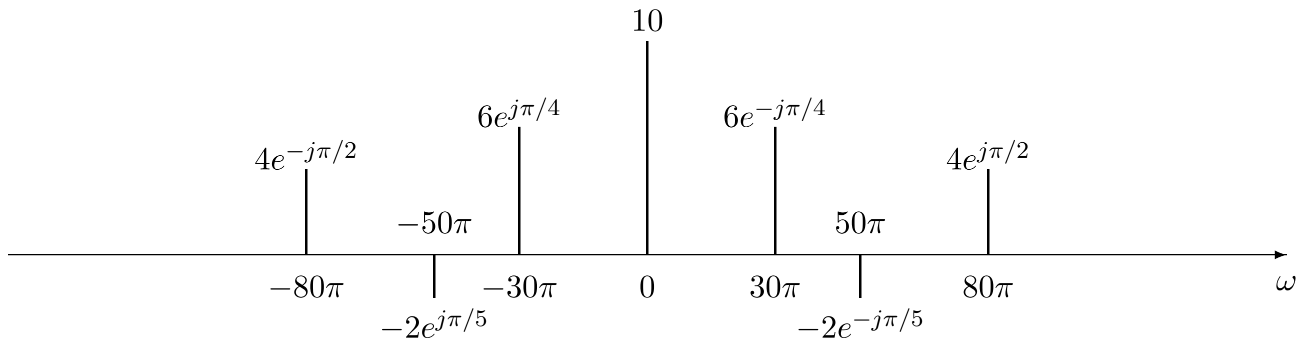
$$A\cos(2\pi\omega t + \phi) = \frac{Z}{2}e^{j2\pi\omega t} + \frac{Z^*}{2}e^{-j2\pi\omega t}$$

where $Z \equiv Ae^{j\phi}$ and Z^* is its conjugate. In other words, we must have both a positive frequency term (the first term on the right of the "=") and a negative frequency term (the second term on the right of the "=") in order to cancel out the imaginary parts of the two terms leaving only the real cosine function. When a number of these pairs of terms are summed, all of their imaginary parts cancel out individually, resulting in a real signal.

(c) Plot the spectrum of the signal $y(t) = 2x(t) - 4\cos(50\pi(t - 0.004))$.
We have to double all the existing spectrum components, because we have $2x(t)$. In addition, the new term $-4\cos(50\pi(t - 0.004))$ is the real part of

$$-4e^{j50\pi(t-0.004)} = -4e^{j(50\pi t+50\pi(-0.004))} = -4e^{j(50\pi t-0.2\pi)} = 4e^{j\pi}e^{j(50\pi t-0.2\pi)} = 4e^{j(50\pi t+0.8\pi)}$$

The spectrum diagram is drawn below using the term, $\frac{1}{2}(-4)e^{j(50\pi t-0.2\pi)}$, but it could also be drawn with an amplitude of $+2$ and a phase of $+0.8\pi$ using the rightmost term above. (Remember that


10
$6e^{j\pi/4}$
$6e^{-j\pi/4}$
$4e^{-j\pi/2}$
$4e^{j\pi/2}$
-50π
50π
-80π
-30π
0
30π
80π
ω
$-2e^{j\pi/5}$
$-2e^{-j\pi/5}$

PROBLEM 2.4:

The two-sided spectrum of a signal $x(t)$ is given in the following table:

frequency (ω)	*complex phasor*
-150π	X_{-2}
-90π	$3e^{j\pi/4}$
0	5
ω_1	X_1
150π	$1+\sqrt{3}j$

(a) If $x(t)$ is a real signal, what are X_1, X_{-2}, and ω_1?

Use conjugate symmetry, $X_{-k} = X_k^*$ to get:

$$\begin{aligned} X_1 &= X_{-1}^* = 3e^{-j\pi/4} \\ X_{-2} &= X_2^* = 1-\sqrt{3}j = 2e^{-j\pi/3} \\ \omega_1 &= -\omega_{-1} = 90\pi \end{aligned}$$

(b) Write an expression for $x(t)$ involving only real numbers and cosine functions.

Take the magnitude and phase from the positive frequency components:

$$x(t) = 5 + 6\cos(90\pi t - \pi/4) + 4\cos(150\pi t + \pi/3)$$

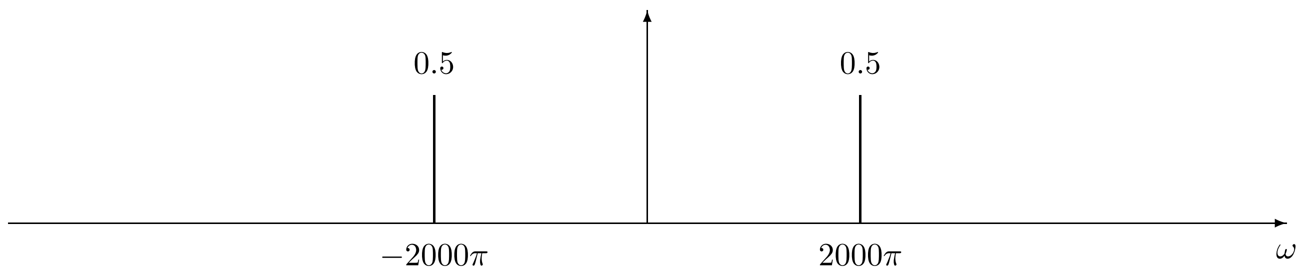
For example, at $\omega = 150\pi$, the complex amplitude is $X_2 = 2e^{j\pi/3}$, so we double the magnitude, $2|X_2|$, and use the phase directly to define the third term in the sum for $x(t)$ above.

PROBLEM 2.5:
In AM radio, the transmitted signal is voice (or music) mixed with a carrier signal. *The carrier is a sinusoid at the assigned broadcast frequency of the AM station. For example, WSB in Atlanta has a* carrier frequency *of 750 kHz. If we use the notation $v(t)$ to denote the voice/music signal, then the actual transmitted signal for WSB might be:*

$$x(t) = (v(t) + A)\cos(2\pi(750 \times 10^3)t)$$

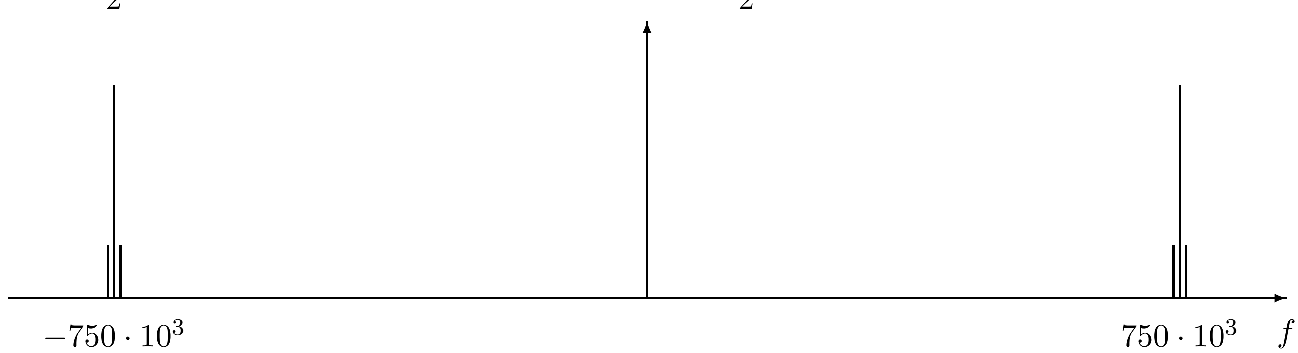
where A is a constant. (A is introduced to make the AM receiver design easier, in which case A must be chosen to be larger than the maximum value of $v(t)$.)

Voice-band signals tend to contain frequencies less than 4000 Hz (4 kHz). Suppose that $v(t)$ is a 1 kHz sinusoid, $v(t) = \cos(2\pi(1000)t)$. Draw the spectrum for $v(t)$.



Now draw the spectrum for $x(t)$, assuming a carrier at 750 kHz. Use $v(t)$ from part (a) and assume that $A = 2$. Hint: Substitute for $v(t)$ and expand $x(t)$ into a sum of cosine terms of three different frequencies.

$$\begin{aligned} x(t) &= [\cos(2 \cdot 10^3\pi t) + 2]\cos(1500 \cdot 10^3\pi t) \\ &= [0.5e^{j2\cdot 10^3\pi t} + 0.5e^{-j2\cdot 10^3\pi t} + 2][0.5e^{j1500\cdot 10^3\pi t} + 0.5e^{-j1500\cdot 10^3\pi t}] \\ &= \frac{1}{4}e^{j1498\cdot 10^3\pi t} + \frac{1}{4}e^{-j1498\cdot 10^3\pi t} + e^{j1500\cdot 10^3\pi t} + e^{-j1500\cdot 10^3\pi t} + \frac{1}{4}e^{j1502\cdot 10^3\pi t} + \frac{1}{4}e^{-j1502\cdot 10^3\pi t} \\ &= \frac{1}{2}\cos[2\pi(749 \cdot 10^3)t] + 2\cos[2\pi(750 \cdot 10^3)t] + \frac{1}{2}\cos[2\pi(751 \cdot 10^3)t] \end{aligned}$$



How would the spectrum of the AM radio signal change if the carrier frequency is changed to 680 kHz (WCNN) and $v(t)$ and A are the same as defined in parts (a) and (b).

The carrier frequency would shift and thus the two sets of spectral components would move from being centered at −750kHz and 750kHz to being centered at −680kHz and 680kHz. The spacing of the components within each set (±1kHz) and the height of the peaks (0.25, 1, and 0.25) would