GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #3

Assigned: 10 Sept 1999
Due Date: 17 Sept 1999 (FRIDAY)

Problem 3.1

(a)

$$\begin{aligned} x(t) &= A\cos[2\pi(f-\Delta)t] + B\cos[2\pi(f\Delta)t] \\ &= \Re e\left\{Ae^{j2\pi(f-\Delta)t} + Be^{j2\pi(f+\Delta)t}\right\} \end{aligned}$$

Therefore,

$$= Ae^{j2\pi(f-\Delta)t} + Be^{j2\pi(f+\Delta)t}$$

(b)

$$\begin{aligned} x(t) &= \Re e\left\{Ae^{j2\pi(f-\Delta)t} + Be^{j2\pi(f+\Delta)t}\right\} \\ &= \Re e\left\{e^{j2\pi ft}\left(Ae^{-j2\pi\Delta t} + Be^{j2\pi\Delta t}\right)\right\} \\ &= \Re e\left\{e^{j2\pi ft}\left(\frac{A+B}{2}\left(e^{-j2\pi\Delta t} + e^{j2\pi\Delta t}\right) + \frac{A-B}{2}\left(e^{-j2\pi\Delta t} - e^{j2\pi\Delta t}\right)\right)\right\} \\ &= \Re e\left\{e^{j2\pi ft}\left((A+B)\cos(2\pi\Delta t) + \frac{A-B}{j}\sin(2\pi\Delta t)\right)\right\} \\ &= \Re e\left\{(\cos(2\pi ft) + j\sin(2\pi ft))\left((A+B)\cos(2\pi\Delta t) + \frac{A-B}{j}\sin(2\pi\Delta t)\right)\right\} \end{aligned}$$

At this point, we multiply these two expressions, and are left with four terms: two are real, and two are imaginary. Keeping the two real terms we get

$$x(t) = (A+B)\cos(2\pi ft)\cos(2\pi\Delta t) + (A-B)\sin(2\pi ft)\sin(2\pi\Delta t).$$

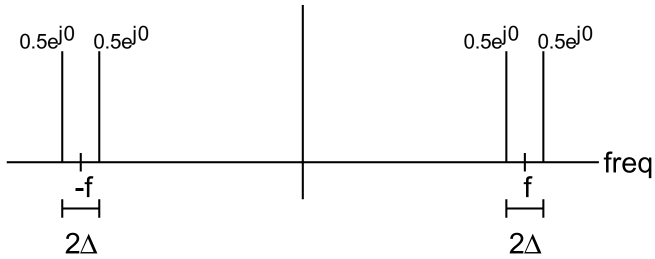
From this expression, we see that $C = A+B$, and $D = A-B$.

If we want

$$x(t) = 2\cos(2\pi ft)\cos(2\pi\Delta t),$$

then $A+B=2$, and $A-B=0$. The solution is $A=B=1$.

We show the spectrum of this signal below.



(c)

x(t) could be periodic if there is a common factor between f and Δ. The resulting frequency would be the greatest common factor between these two numbers. In other words, the ratio of f and Δ must be a rational number.

Another way to do part (b) that might be a little more straightforward:

$$\begin{aligned}
x(t) &= \Re e\left\{Ae^{j2\pi(f-\Delta)t} + Be^{j2\pi(f+\Delta)t}\right\} \\
&= \Re e\left\{e^{j2\pi ft}\left(Ae^{-j2\pi\Delta t} + Be^{j2\pi\Delta t}\right)\right\} \\
&= \Re e\left\{(\cos(2\pi ft) + j\sin(2\pi ft))(A\cos(2\pi\Delta t) - jA\sin(2\pi\Delta t) + B\cos(2\pi\Delta t) + jB\sin(2\pi\Delta t))\right\} \\
&= \Re e\left\{A\cos(2\pi\Delta t)\cos(2\pi ft) + B\cos(2\pi ft)B\cos(2\pi\Delta t) - j^2A\sin(2\pi ft)\sin(2\pi\Delta t) + \ldots\right. \\
&\qquad \left. j^2B\sin(2\pi ft)\sin(2\pi\Delta t) + j(\cdots)\right\}
\end{aligned}$$

The terms in the imaginary part don't matter, and $j^2 = -1$, so we get:

$$x(t) = (A+B)\cos(2\pi ft)\cos(2\pi\Delta t) + (A-B)\sin(2\pi ft)\sin(2\pi\Delta t)$$

and then we can see that $C = A + B$, and $D = A - B$.

Problem 3.2

Left graph	Right graph	Comments
(a)	**2**	Graph is $-\sin(3\pi t)$
(b)	**(4)**	Two frequencies and periodic at 0.5Hz.
(c)	**5**	The only graph with a DC (zero frequency) component
(d)	**(3)**	Graph is very close to $\sin(3\pi t)$
(e)	**(1)**	Two frequencies and and periodic at 0.3Hz

Problem 3.3

(a), (b)

The equation for the graph is

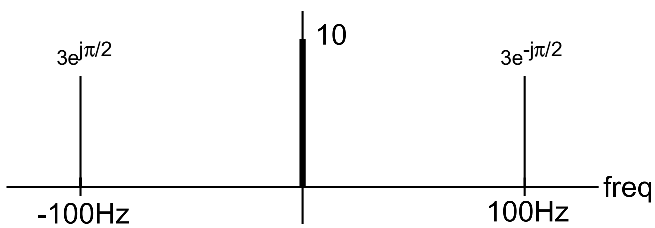
$x(t) = 6\sin(200\pi t) + 10 = 10 + 6\cos(200\pi t - \pi/2)$

Frequency of the DC component: 0Hz

Frequency of the cosine component: $T = 0.01s \rightarrow\ f = 100Hz$.

(c)

$$x(t) = 10 + 6\cos(200\pi t - \pi/2)$$
$$= 10 + 3e^{j(200\pi t)}e^{-j\pi/2} + 3e^{j(-200\pi t)}e^{j\pi/2}$$



Problem 3.4

(a) We have twelve notes per octave that are equally spaced, and the frequency of one note in an octave is exactly twice that of the frequency of the same note in the next lower octave. The only way to achieve this effect is by using an exponential spacing in frequency.

$$(2^x)^1 2 = 2 \rightarrow, x = 1/12$$

.

Therefore the spacing must be $2^{1/12}$

As a side note, the reason this frequency spacing works for those listening to this scale is that the biological sensing of hearing utilizes a exponential spacing.

(b)

note name	C	D^b	D	E^b	E	F	G^b	G	A^b	A	B^b	B	C
note number	40	41	42	43	44	45	46	47	48	49	50	51	52
frequency	261.6	277.2	293.7	311.1	329.6	349.2	370.0	392.00	415.3	440.0	466.2	493.9	523.3

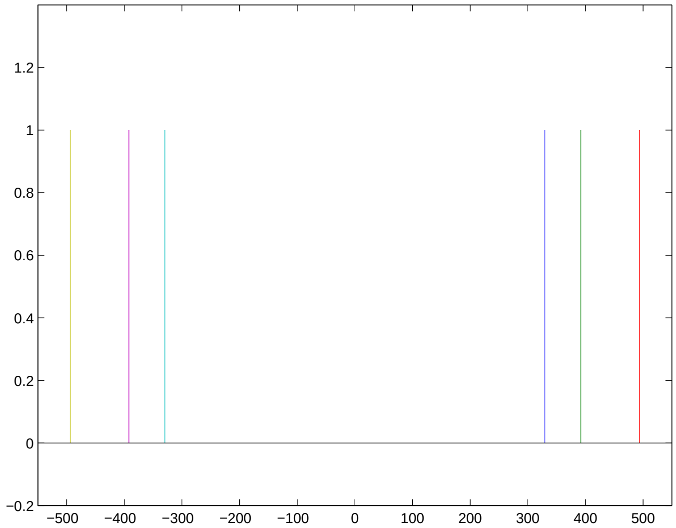
Observe that Note number 52 is twice the frequency of Note number 40.

(c) We know that A above middle C is note number 49, and is at a frequency of 440Hz. We also know that each additional note will be a multiplicative factor, $2^{1/12}$, increase in frequency.

Therefore, $f = 440 \times 2^{(n-49)/12}$

(d)

An E-Minor cord: E = 329.6Hz, G = 392Hz, B = 493.9Hz



Problem 3.5

(a)

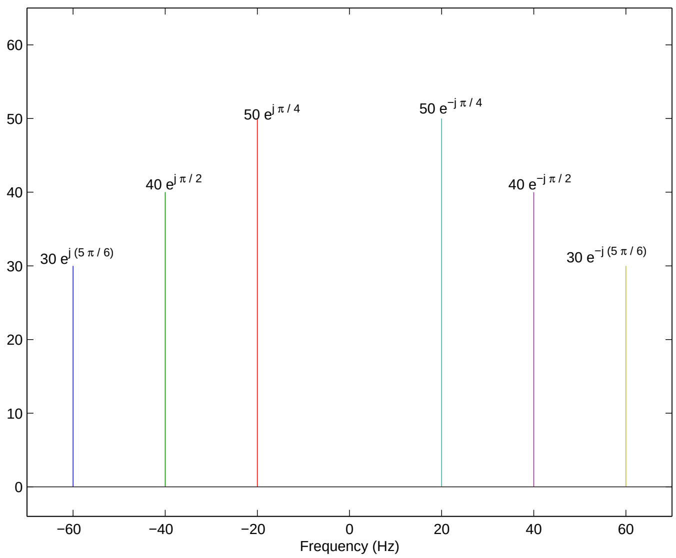
x(t) has the following three components:

1. Amplitude = 100, frequency = 20Hz, and phase = $-\pi/4$.

2. Amplitude = 80, frequency = 40Hz, and phase = $-\pi/2$.

3. Amplitude = 60, frequency = 60Hz, and phase = $-5\pi/6$.

We show the resulting spectrum below.



MATLAB code used to generate this plot:

```
y2 = [ [-60 -40 -20 20 40 60]' [-60 -40 -20 20 40 60]'];
x2 = [ [0 0 0 0 0 0]' [30 40 50 50 40 30]'];
plot(y2',x2',[-70 70],[0 0],'k-');
axis([-70 70 -4 65]);
xlabel('Frequency (Hz)');
```

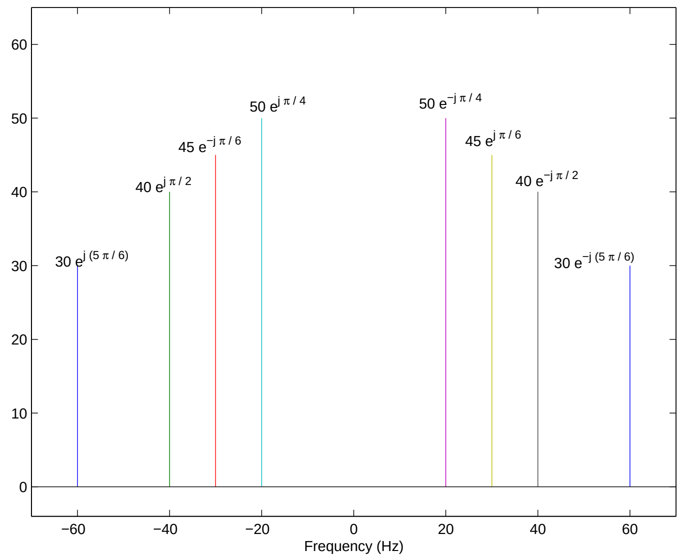
(b) This signal is periodic, with a period = 50ms (20Hz). We have the first, second, and third harmonics present.

(c) This new signal has an additional component:

Amplitude = 90, frequency = 30Hz, and phase = $\pi/6$.

The new period for this signal is 100ms, corresponding to a fundamental frequency of 10Hz. We now have the second, third, fourth, and sixth harmonics present.

The spectrum plot of this new signal.



Code used to generate this plot: `y3 = [ [-60 -40 -30 -20 20 30 40 60]' [-60 -40 -30 -20 20 30 40 60]'];`

```
x3 = [ [0 0 0 0 0 0 0 0]' [30 40 45 50 50 45 40 30]'];
plot(y3',x3',[-70 70],[0 0],'k-'); axis([-70 70 -4 65]);
xlabel('Frequency (Hz)');
```

(d) Ths new signal has an additional component:

Amplitude = 10, frequency = $140/\pi$ Hz, and phase = $\pi/2$.

This signal is not periodic.

Problem 3.6

(a)

First we derive for the instantaneous frequency as

$$\omega_i(t) = \frac{d}{dt}\Psi(t) = \frac{d}{dt}\left(\alpha t^2 + \beta t + \phi\right) = 2\alpha t + \beta.$$

Substituting into this equation and noting that the chirp starts at $t = 0$ and ends at $t = T_2$, we get the beginning instantaneous frequency (ω_1), and the ending instantaneous frequency (ω_2) as

$$\omega_1 = \beta$$

$$\omega_2 = 2\alpha T_2 + \beta$$

(b)

$$x(t) = \cos(2\pi(25t^2 - 25t))$$

As a result,

$$\omega_i(t) = \frac{d}{dt}(2\pi(25t^2 - 25t)) = 2\pi(50t - 25)$$

(c)

It seems that instantaneous frequency might become negative. However, since the cosine is made up of both a positive frequency component and a negative frequency component via the inverse Euler formula, the negative frequency complex exponential also contributes to the instantaneous frequency (the dashed red line in the figure below—left-hand side). If the question is interpreted as asking how the frequency is perceived then the plot on the right-hand side of the figure below is the correct one, because humans cannot hear negative frequencies.

Plot of instantaneous frequency for 1 sec: