GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #4

SOLUTION

Date: 24 Sept 99 (FRIDAY)

PROBLEM 4.1:

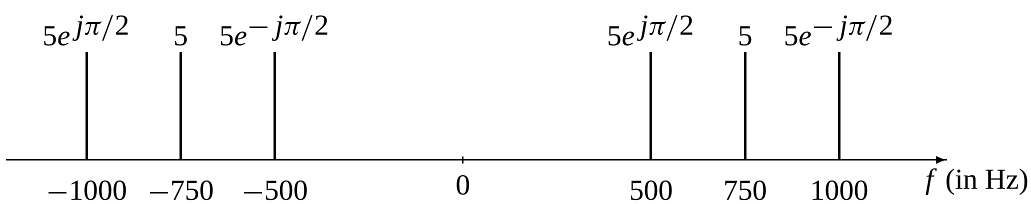
Consider the signal

$$x(t) = [10 + 20\sin(500\pi t)]\cos(1500\pi t)$$

(a) Sketch the two-sided spectrum of this signal. Be sure to label important features of the plot. *Hint: Recall the AM spectrum from a previous homework set.*

Use the inverse Euler formulae:

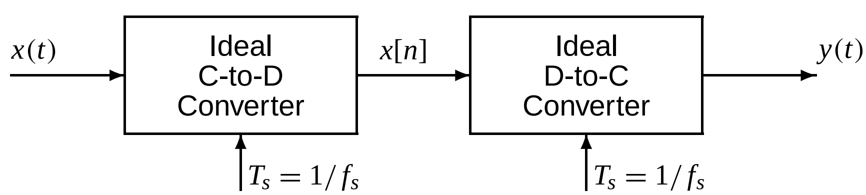
$$\begin{aligned} x(t) &= [10 + 20\sin(500\pi t)]\cos(1500\pi t) \\ &= [10 + (10e^{-j500\pi t + j\pi/2} + 10e^{j500\pi t - j\pi/2})](\tfrac{1}{2}e^{-j1500\pi t} + \tfrac{1}{2}e^{j1500\pi t}) \\ &= (5e^{-j1500\pi t} + 5e^{j1500\pi t}) + (5e^{-j2000\pi t + j\pi/2} + 5e^{j2000\pi t - j\pi/2}) + (5e^{-j1000\pi t - j\pi/2} + 5e^{j1000\pi t + j\pi/2}) \end{aligned}$$



(b) Is this waveform periodic? If so, what is the period?

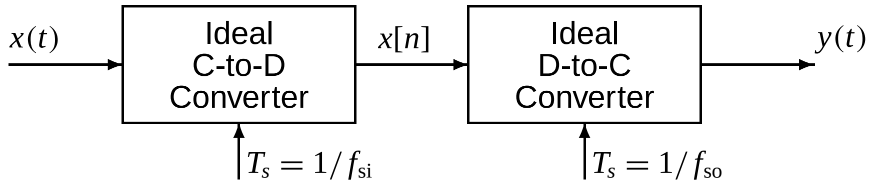
Yes. The three cyclic frequencies (500, 750, and 1000 Hz) are all integral multiples of $f_0 = 250$ Hz (i.e. 250 is the greatest common factor of 500, 750, and 1000). Thus, the period of the cyclic waveform is $T_0 = 1/f_0 = 4$ ms.

(c) What is the minimum sampling rate f_s that can be used in the following system so that $y(t) = x(t)$?



The sampling rate f_s must be greater than twice the highest frequency in the signal. The highest-frequency component of $x(t)$ is 1000 Hz, and thus $f_s > 2000$ Hz.

PROBLEM 4.2:



(a) Suppose that the discrete-time signal $x[n]$ is given by the formula

$$x[n] = 10\cos(0.25\pi n - \pi/4)$$

If the sampling rate of the C-to-D converter is $f_{\text{si}} = 2000$ samples/second, many *different* continuous-time signals $x(t) = x_\ell(t)$ could have been inputs to the above system. Determine two such inputs with frequency less than 2000 Hz; i.e., find $x_1(t)$ and $x_2(t)$ such that $x[n] = x_1(nT_{\text{si}}) = x_2(nT_{\text{si}})$ if $T_{\text{si}} = 1/2000$.

We can determine $x_1(t)$ using direct substitution as follows:

$$\begin{aligned} x[n] &= x_1(nT_{\text{si}}) \\ 10\cos(0.25\pi n - \pi/4) &= A\cos(2\pi f_1 n T_{\text{si}} + \phi_1) \end{aligned}$$

Using $T_{\text{si}} = 1/2000$, we can solve for $A = 10$, $f_1 = 250$ Hz, and $\phi_1 = -\pi/4$ to give

$$x_1(t) = 10\cos(2\pi \cdot 250t - \pi/4)$$

To determine another input, x_2, that would give the same sampled result, we must find an aliased input which adheres to the following equation

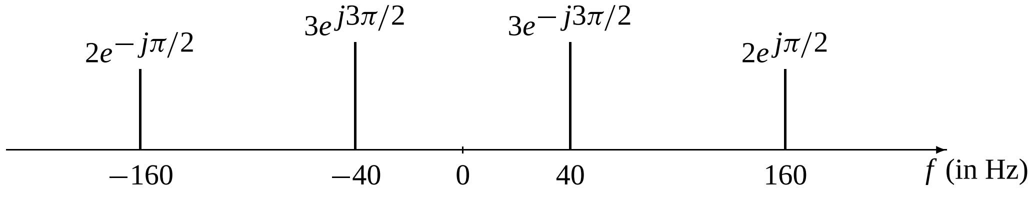
$$x_2(t) = 10\cos(2\pi \cdot (250 + \ell f_{\text{si}})t - \pi/4)$$

where ℓ is an integer. For $f_{\text{si}} = 2000$ samples/second, the only value of ℓ that gives a resulting frequency $f_2 < 2000$ Hz is $\ell = -1$, which gives the following input waveform

$$\begin{aligned} x_2(t) &= 10\cos(2\pi \cdot (250 + (-1) \cdot 2000)t - \pi/4) \\ &= 10\cos(-2\pi \cdot 1750t - \pi/4) \\ &= 10\cos(2\pi \cdot 1750t + \pi/4) \end{aligned}$$

where the final step is a result of the fact that $\cos(\theta) = \cos(-\theta)$ (i.e. cosine is an even function). Note that input $x_2(t)$ results in folding, which causes the phase of the resulting signal to invert (from $\pi/4$ in the input to $-\pi/4$ in the sampled signal).

(b) Now if the input $x(t)$ is given by the two-sided spectrum representation shown below,



Determine the spectrum for $x[n]$ when $f_{\text{si}} = 120$ samples/sec. Make a plot for your answer, but label the frequency, amplitude and phase of each spectral component.

From the spectrum

$$
\begin{aligned}
x(t) &= 6\cos(2\pi\cdot 40t - 3\pi/2) + 4\cos(2\pi\cdot 160t + \pi/2) \\
&= 6\cos(2\pi\cdot 40t + \pi/2) + 4\cos(2\pi\cdot 160t + \pi/2)
\end{aligned}
$$

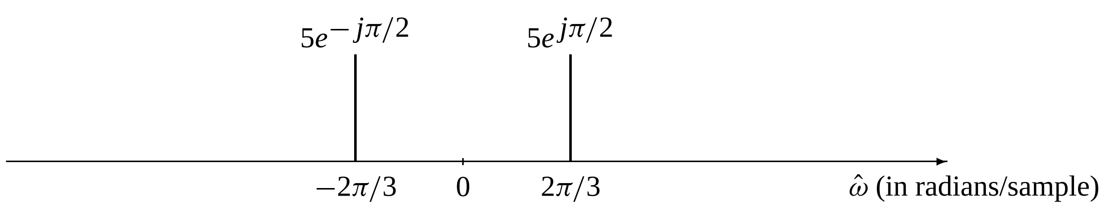
By replacing t with $nT_{\text{si}} = n/120$, we derive

$$x[n] = 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{8\pi}{3}n + \pi/2\right)$$

From this equation, the normalized frequencies of the two terms are $\hat{\omega}_1 = 2\pi/3$ radians/sample and $\hat{\omega}_2 = 8\pi/3$ radians/sample. $\hat{\omega}_1 < \pi$, and thus it does not alias. However, $\hat{\omega}_2 >= \pi$, and thus it does alias. By subtracting 2π from $\hat{\omega}_2$, we find that it aliases to a nomalized frequency of $2\pi/3$ radians/sample. In other words, the two analog frequencies give the same sampled frequency, resulting in them effectively summing, as shown in the following equations:

$$
\begin{aligned}
x[n] &= 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{8\pi}{3}n + \pi/2\right) \\
&= 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{2\pi}{3}n + \pi/2\right) \\
&= 10\cos\left(\frac{2\pi}{3}n + \pi/2\right)
\end{aligned}
$$

The resulting spectrum plot is



(c) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\text{so}} = 120$ Hz.

$$f = \frac{\hat{\omega}}{2\pi}\cdot f_{\text{so}} = \frac{1}{3}\cdot 120 = 40\text{Hz}$$

$$x(t) = 10\cos(80\pi t + \pi/2)$$

(d) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\text{so}} = 200$ Hz. In other words, the sampling rates of the two converters are different.

$$f = \frac{\hat{\omega}}{2\pi}\cdot f_{\text{so}} = \frac{1}{3}\cdot 200 \approx 66.67\text{Hz}$$

$$x(t) = 10\cos\left(\frac{400\pi}{3}t + \pi/2\right)$$

PROBLEM 4.3:

In the rotating disk and strobe demo described in Chapter 4 of *DSP First*, we observed that different flashing rates of the strobe light would make the spot on the disk stand still.

(a) Assume that the disk is rotating in the counter-clockwise direction at a constant speed of 600 rpm (revolutions per minute). Express the movement of the spot on the disk as a rotating complex phasor.

We define the plane of the disk to be the complex plane. The position of the dot in rectangular coordinates is (x, y), and the polar coordinates are r (the distance from the center of the disk to the center of the dot) and θ (the angle from the positive x axis). The rotation rate is constant (600 RPM or 10 rotations/second), and thus the angle of the spot changes linearly defined by $\theta = \omega t + \phi$. Given these pieces of information and the fact that $\omega = 2\pi \cdot 10$ radians/second, we can define the location of the center of the dot, $z = (x, y)$ as follows:

$$z(t) = re^{j20\pi t + j\phi}$$

which gives the following coordinates for the point

$$\begin{aligned} x(t) &= r\cos(20\pi t + \phi) \\ y(t) &= r\sin(20\pi t + \phi) \end{aligned}$$

(b) If the strobe light can be flashed at a rate of n flashes *per second* where n is an integer greater than zero, determine all possible flashing rates such that the disk can be made to stand still.
NOTE: the only possible flashing rates are integers: 1 per second, 2 per second, 3 per second, etc.

When $n = 10$, the strobe flashes exactly once every disk rotation, apparantly making the dot look like it is standing still. Assuming that the flashing is defined such that one of the flashes occurs at $t = 0$, the location of the dot on this disk is at angle $\theta = \phi$.

When $n < 10$, the dot can also be made to stand still if the strobe flashes exactly once every m revolutions, where m is an integer. From these terms, $m \cdot n = 10$. Given that m and n must both be integers, the values of n (and, subsequently, m) must be the factors of 10, or $n = 1, 2, 5$.

When $n > 10$, the dot(s) seems to stand still when $n = 10m$, where m is an integer (i.e. $n = 20, 30, \ldots$). In each of these cases, the number of dots that appears to be standing still is equal to m. For example, if $n = 30$, the strobe flashes exactly three times during each rotation of the disk, and thus three dots separated by $2\pi/3$ radians appear to stand still on the wheel.

(c) If the flashing rate is 11 times per second, explain how the spot will move and write a complex phasor that gives the position of the spot at each flash.

The sampling by the strobe results in the following equation:

$$z[n] = z(n/11) = re^{j\phi}e^{j\frac{20\pi}{11}n}$$

The resulting sampled radian frequency is $20\pi/11$, which aliases to $\hat{\omega} = -2\pi/11$ (by subtracting $2\pi = 22\pi/11$ from the sampled frequency). The resulting discrete-time signal is

$$z[n] = re^{j\phi}e^{-j\frac{2\pi}{11}n}$$

Given that the strobe flashes at 11 flashes/second, $\hat{\omega} = -2\pi/11$ radians/flash represents an apparant rotation frequency of $\omega = -2\pi$ radians/second, or 1 *clockwise* rotation/second or 60 RPM. (Note: The rotation is clockwise because the rotation frequency is negative.)

(d) Draw a spectrum plot of the discrete-time signal in part (c) to explain your answer.

The spectrum repeats every 2π, so there are lines at $\hat{\omega} = 20\pi/11 + 2\pi\ell$, where $\ell = 0, \pm 1, \pm 2, \pm 3, \ldots$