GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #7

Assigned: 10 September 1999
Due Date: 17 September 1999 (FRIDAY)

PROBLEM 7.1:

We now have four ways of describing an LTI system: the difference equation; the impulse response, $h[n]$; the frequency response, $H(e^{j\hat{\omega}})$; and the system function, $H(z)$. In the following, you are given one of these representations and you must find the other three.

(a) $y[n] = \frac{1}{4}(x[n] - x[n-4])$.

$$h[n] = \tfrac{1}{4}\left(\delta[n] - \delta[n-4]\right)$$

$$H(z) = \tfrac{1}{4}\left(1 - z^{-4}\right)$$

$$H(e^{j\hat{\omega}}) = \tfrac{1}{4}\left(1 - e^{-j4\hat{\omega}}\right) = \tfrac{1}{4}e^{-j2\hat{\omega}}(e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}) = \frac{j}{2}\sin(2\hat{\omega})e^{-j2\hat{\omega}}$$

(b) $h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$.

$$y[n] = x[n] + 2x[n-1] + 3x[n-2] + 2x[n-3] + x[n-4]$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$\begin{aligned} H(e^{j\hat{\omega}}) &= 1 + 2e^{-j\hat{\omega}} + 3e^{-j2\hat{\omega}} + 2e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}} \\ &= e^{-j2\hat{\omega}}\left(3 + 2e^{j\hat{\omega}} + 2e^{-j\hat{\omega}} + e^{j2\hat{\omega}} + e^{-j2\hat{\omega}}\right) \\ &= e^{-j2\hat{\omega}}\left(3 + 4\cos(\hat{\omega}) + \cos(2\hat{\omega})\right) \end{aligned}$$

$$\text{or,} \quad H(e^{j\hat{\omega}}) = \frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}} = \frac{e^{-j2.5\hat{\omega}}(e^{j2.5\hat{\omega}} - e^{-j2.5\hat{\omega}})}{e^{-j0.5\hat{\omega}}(e^{j0.5\hat{\omega}} - e^{-j0.5\hat{\omega}})} = e^{-j2\hat{\omega}}\left(\frac{\sin(2.5\hat{\omega})}{\sin(0.5\hat{\omega})}\right)$$

(c) $H(e^{j\hat{\omega}}) = [2 + 2\cos(\hat{\omega})]e^{-j\hat{\omega}2}$.

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}}$$

$$h[n] = \delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

$$y[n] = x[n-1] + 2x[n-2] + x[n-3]$$

$$H(z) = z^{-1} + 2z^{-2} + z^{-3}$$

(d) $H(z) = z^{-3} + z^{-6} + z^{-9}$.

$$y[n] = x[n-3] + x[n-6] + x[n-3]$$

$$h[n] = \delta[n-3] + \delta[n-6] + \delta[n-3]$$

$$H(e^{j\hat{\omega}}) = e^{-j3\hat{\omega}} + e^{-j6\hat{\omega}} + e^{-j9\hat{\omega}} = e^{-j6\hat{\omega}}\left(1 + 2\cos(3\hat{\omega})\right)$$

PROBLEM 7.2:

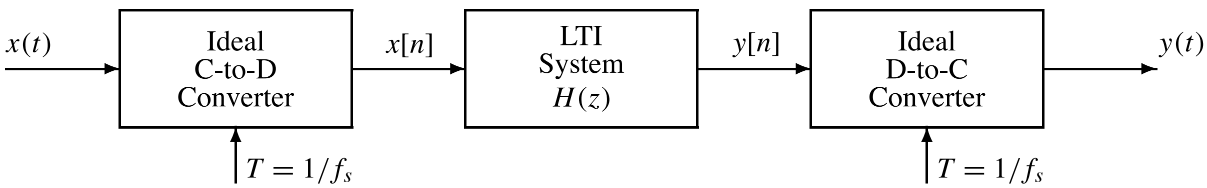
The input to the C-to-D converter in the figure below is

$$x(t) = 3 + 2\cos(6000\pi t - \pi/4) + 11\cos(12000\pi t - \pi/3)$$

The system function for the LTI system is

$$H(z) = \frac{1}{4}(1 - z^{-4})$$

If $f_s = 8000$ samples/second, determine an expression for $y(t)$, the output of the D-to-C converter.



This is Problem 7.3 of Spring 1999. Try working it before you consult the answer.

$$H(z) = \frac{1}{4}\left(1 - z^{-1}\right)$$

$$H(e^{j\hat{\omega}}) = \tfrac{1}{4}(1 - e^{-j4\hat{\omega}}) = \tfrac{1}{4}(e^{j2\hat{\omega}} - e^{-j2\hat{\omega}})e^{-j2\hat{\omega}} = \frac{j}{2}\sin(2\hat{\omega})e^{-j2\hat{\omega}} = \tfrac{1}{2}\sin(2\hat{\omega})e^{j(\pi/2)}e^{-j2\hat{\omega}}$$

Since $f_s = 8000$ samples / second, we get the discrete-time cosines for $x[n]$:

$$x[n] = 3 + 2\cos(\tfrac{3}{4}\pi n - \pi/4) + 11\cos(\tfrac{6}{4}\pi n - \pi/3)$$

$$= 3 + 2\cos(\tfrac{3}{4}\pi n - \pi/4) + 11\cos(-\tfrac{2}{4}\pi n - \pi/3)$$

$$= 3 + 2\cos(\tfrac{3}{4}\pi n - \pi/4) + 11\cos(\tfrac{1}{2}\pi n + \pi/3)$$

because we can always add a factor of $2\pi n$, which is the source of folding.

Next we want to compute y[n], which is $h[n] * x[n]$. Since we have a sum of sinusoids, *we don't need to do the convolution*; instead, we only need to evaluate the frequency response at each of these frequencies:

At $\hat{\omega} = 0$, $H(e^{j\hat{\omega}}) = \tfrac{1}{2}\sin(0)e^{j(\pi/2)} = 0$.

At $\hat{\omega} = \tfrac{3}{4}\pi$, $H(e^{j\hat{\omega}}) = \tfrac{1}{2}\sin(\tfrac{3}{2}\pi)e^{j(\pi/2)}e^{-j\frac{3}{2}\pi} = \tfrac{1}{2}(-1)e^{-j\pi} = \tfrac{1}{2}$.

At $\hat{\omega} = \tfrac{1}{2}\pi$, $H(e^{j\hat{\omega}}) = \tfrac{1}{2}\sin(\pi)e^{-j(\pi/2)}e^{-j2\pi} = 0$.

Therefore, we get

$$y[n] = (\tfrac{1}{2})2\cos(\tfrac{3}{4}\pi n - \pi/4) = \cos(\tfrac{3}{4}\pi n - \pi/4)$$

PROBLEM 7.3:

Consider the following MATLAB program:

```
nn = 0:16000;
xx = 3 + 2*cos(0.75*pi*nn-pi/4) + 11*cos(1.5*pi*nn-pi/3);
yy = conv([1,0,0,0,-1]/4,xx);
soundsc(yy,8000)
```

(a) What is the system function $H(z)$ of the system that is implemented by the `conv( )` statement?
The `conv( )` statement represents the transfer function

$$H(z) = \tfrac{1}{4}\left(1 - z^{-4}\right)$$

(b) What is the frequency response of the system?
From Problem 7.2, the frequency response is $H(e^{j\hat{\omega}}) = \frac{1}{2}\sin(2\hat{\omega})e^{j\pi/2}e^{-j2\hat{\omega}}$

(c) Neglecting the end effects in the convolution, determine $y(t)$ that describes the signal produced by the `soundsc( )` statement. *Hint: The result of Problem 7.2 should be useful here.*

From Problem 7.2, the final solution for $y(t)$ is

$$y[n] = \cos(\tfrac{3}{4}\pi n - \pi/4)$$

Now we use the sampling theorem to convert the discrete-time frequency of $\hat{\omega} = 3\pi/4$ to a frequency in hertz via:

$$f = \frac{\hat{\omega}}{2\pi} f_s = \left(\frac{3\pi/4}{2\pi}\right) 8000 \text{ Hz} = 3000 \text{ Hz}$$

and we get

$$y(t) = \cos(2\pi(3000)t - \pi/4)$$

PROBLEM 7.4:

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) What is the impulse response, $h[n]$, of this system?

$$h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$$

(b) Determine the system function $H(z)$ for this system.

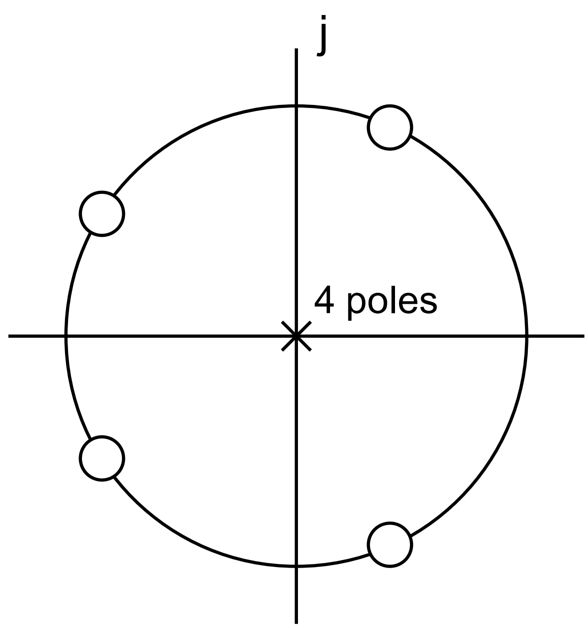
$$H(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right) = \frac{1 - z^{-5}}{5(1 - z^{-1})} = \frac{z^5 - 1}{5z^4(z-1)}$$

(c) Plot the poles and zeros of $H(z)$ in the complex z-plane. *Hint: Remember the N-th roots of unity.*
There is neither a pole or zero at $z = 1$.

Poles: Four at $z = 0$

Zeros: The numerator is $1 - z^{-5}$, therefore, we must solve $z^5 = 1$ and we get $z = e^{j\frac{2\pi n}{5}}$, where $n = 1, 2, 3, 4$. These zeros are on the unit circle and their angles are multiples of $2\pi/5$ or 72°.

We plot the poles and zeros below.



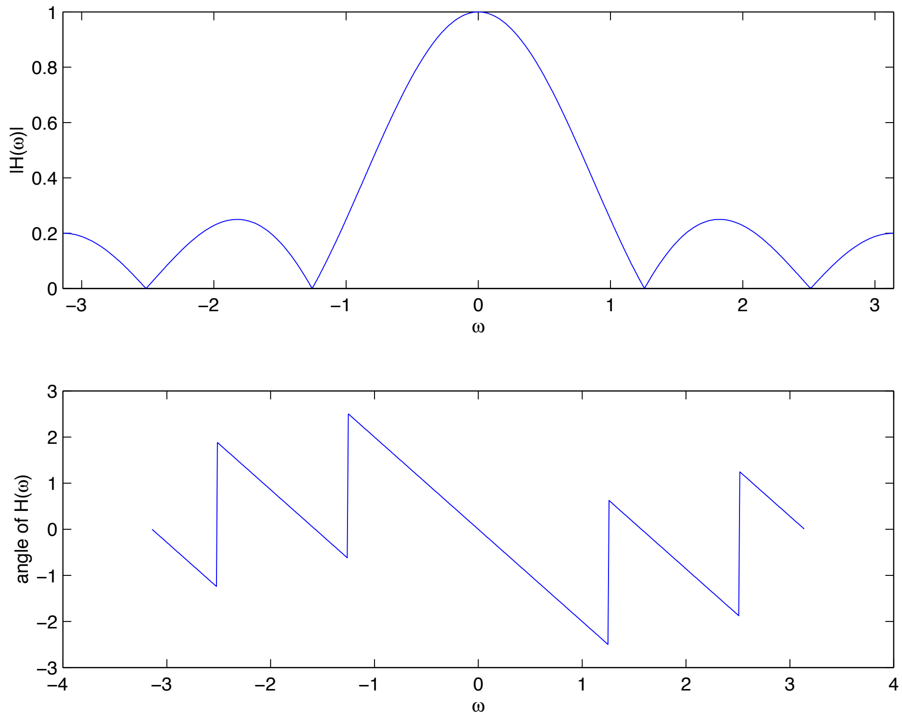
(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.

$$H(e^{j\hat{\omega}}) = \frac{1}{5}\frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}}$$

(e) Show that your answer in (d) can be expressed in the form $H(e^{j\hat{\omega}}) = \dfrac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)}e^{-j2\hat{\omega}}$.

$$H(e^{j\hat{\omega}}) = \frac{1}{5}\frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}} = \frac{1}{5}e^{-j(5/2)\hat{\omega}}e^{j\hat{\omega}/2}\frac{e^{j(5/2)\hat{\omega}} - e^{-j(5/2)\hat{\omega}}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}} = e^{-j2\hat{\omega}}\frac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)}$$

(f) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above (or plot it using `freqz( )`).



MATLAB Code to make this plot:

```
 ww = [-pi:0.01:pi];
 HH = 0.2 * (sin( 5 * ww / 2) ./ sin( ww / 2) ) .* exp( j*(-2)*ww);
%-- or use [HH,ww] = freqz( ones(1,5)/5, 1, ww);
 subplot(2,1,1)
    plot(w,abs(HH));  axis([-pi pi 0 1]);
 subplot(2,1,2),
    plot(w,angle(HH));
```

(g) Suppose that the input is

$$x[n] = 5 + 4\cos(0.1\pi n) + 3\cos(0.4\pi n - \pi/4) \qquad \text{for } -\infty < n < \infty$$

Obtain an expression for the output in the form $y[n] = A + B\cos(\hat{\omega}_0 n + \phi_0)$. (In other words, one of the sinusoids is removed by the filter.)

For this signal, we need to evaluate the frequency response at three frequencies:

At $\hat{\omega} = 0$, $H(e^{j\hat{\omega}}) = e^{-j0}\dfrac{\sin(0(5/2))}{5\sin(0(1/2))} \rightarrow 1$.

At $\hat{\omega} = 0.1\pi$, $H(e^{j\hat{\omega}}) = e^{-j0.2\pi}\dfrac{\sin(0.25\pi)}{5\sin(0.05\pi)} = 0.904e^{-j0.2\pi}$.

At $\hat{\omega} = 0.4\pi$, $H(e^{j\hat{\omega}}) = e^{-j0.8\pi}\dfrac{\sin(\pi)}{5\sin(\pi/5)} = 0$.

Therefore the solution is

$$x[n] = 5(1) + 4(0.904)\cos(0.1\pi n - 0.2\pi) + 3(0)\cos(0.4\pi n - \pi/4) = 5 + 3.616\cos(0.1\pi n - 0.2\pi)$$

PROBLEM 7.5:

The diagram in Figure 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system. In Figure 1,

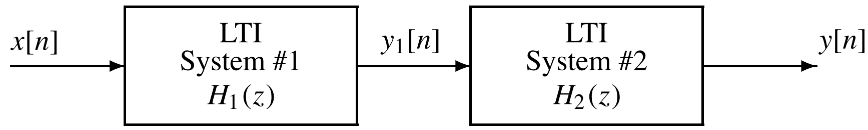


Figure 1: Cascade connection of two LTI systems.

assume that both systems are 3-point moving averagers; i.e.,

$$y_1[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) \quad \text{and} \quad y[n] = \frac{1}{3}(y_1[n] + y_1[n-1] + y_1[n-2]).$$

(a) Determine the system function $H(z) = H_1(z)H_2(z)$ for the overall system.

$$H_1(z) = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right)$$

$$H_2(z) = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right)$$

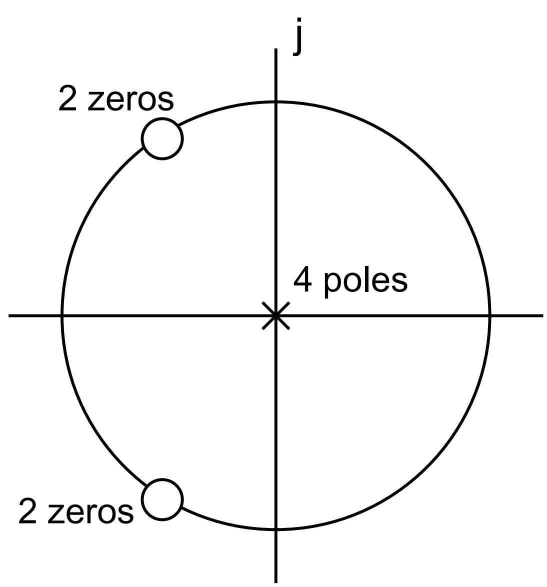
$$H(z) = H_1(z)H_2(z) = \frac{1}{9}\left(1 + z^{-1} + z^{-2}\right)^2$$

$$= \frac{1}{9}\left(1 + z^{-2} + z^{-4} + 2z^{-1} + 2z^{-2} + 2z^{-3}\right) = \frac{1}{9}\left(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}\right)$$

(b) Plot the poles and zeros of $H(z)$ in the z-plane.

$$H(z) = H_1(z)H_2(z) = \frac{1}{9}\left(1 + z^{-1} + z^{-2}\right)^2 = \frac{1}{9}\left(\frac{1 - z^{-3}}{1 - z^{-1}}\right)^2 = \frac{(z^3 - 1)^2}{9z^4(z-1)^2}$$

Thus, we have four poles at $z = 0$, and double zeros at both $z = e^{j2\pi/3}$ and $z = e^{j4\pi/3}$. There is no pole or zero at $z = 1$, because $z = 1$ is a root of both the numerator and the denominator. We plot the poles and zeros below.



(c) Use multiplication of z-transform polynomials to determine the impulse response $h[n]$ of the overall system in Figure 1.

This was already done in part (a). The impulse response is the inverse z-transform:

$$h[n] = \frac{1}{9}\left(\delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]\right)$$

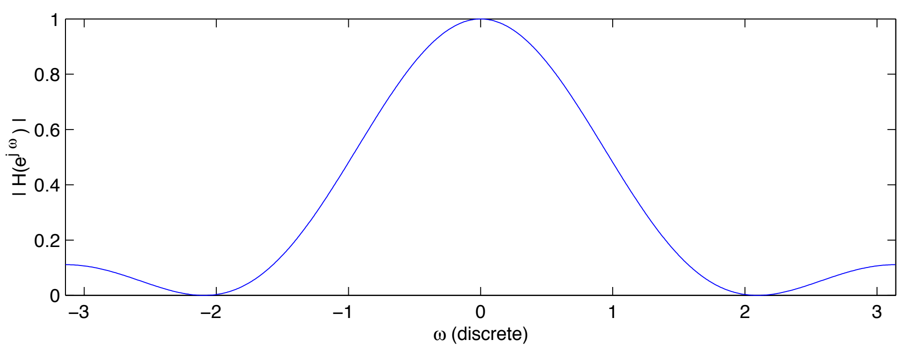
(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of the overall cascade system.

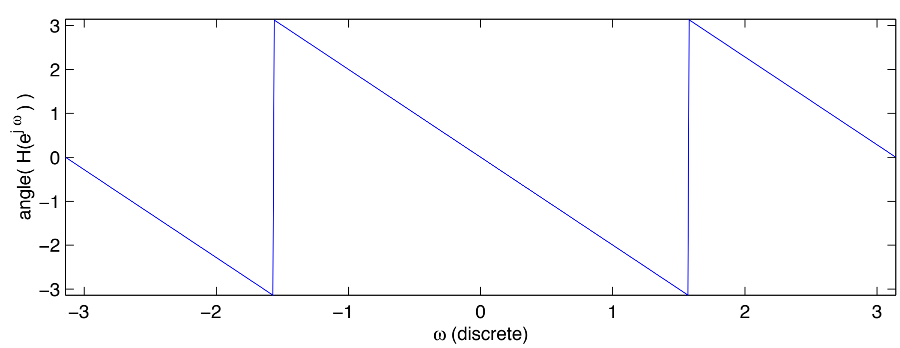
We know from earlier results that the frequency response of the 3-pt averager is: $\frac{1}{3}e^{-j\hat{\omega}}\frac{\sin((3/2)\hat{\omega})}{\sin(\hat{\omega}/2)}$

Therefore, since $H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$, we get

$$H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}}\frac{\sin^2((3/2)\hat{\omega})}{9\sin^2(\hat{\omega}/2)}$$

(e) Use your result from (d) as an aid in sketching the frequency response (magnitude and phase) functions of the overall cascade system for $-\pi \leq \hat{\omega} \leq \pi$.





PROBLEM 7.6:

The system function of a linear time-invariant filter is given by the formula

$$H(z) = (1 - z^{-1})(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1})(1 - 0.5e^{j\pi/3}z^{-1})(1 - 0.5e^{-j\pi/3}z^{-1})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. Make sure that all the filter coefficients $\{b_k\}$ in your difference equation are purely real.
A few simplifications:

$$(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1}) = 1 - \left(e^{j\pi/2} + e^{-j\pi/2}\right)z^{-1} + e^{j\pi/2}e^{-j\pi/2}z^{-2} = 1 + z^{-2}$$

$$(1 - 0.5e^{j\pi/3}z^{-1})(1 - 0.5e^{-j\pi/3}z^{-1}) = 1 - 0.5\left(e^{j\pi/3} + e^{-j\pi/3}\right)z^{-1} + (0.5)^2 e^{j\pi/3}e^{-j\pi/3}z^{-2} = 1 - 0.5z^{-1} + 0.25z^{-2}$$

Therefore,

$$\begin{aligned} H(z) &= (1 - z^{-1})(1 + z^{-2})(1 - 0.5z^{-1} + 0.25z^{-2}) \\ &= (1 - z^{-1} + z^{-2} - z^{-3})(1 - 0.5z^{-1} + 0.25z^{-2}) \\ &= 1 - 1.5z^{-1} + 1.75z^{-2} - 1.75z^{-3} + 0.75z^{-4} - 0.25z^{-5} \end{aligned}$$

The difference equation is

$$y[n] = x[n] - 1.5x[n-1] + 1.75x[n-2] - 1.75x[n-3] + 0.75x[n-4] - 0.25x[n-5]$$

(b) What is the output if the input is $x[n] = \delta[n]$?
The output is the impulse response:

$$h[n] = \delta[n] - 1.5\delta[n-1] + 1.75\delta[n-2] - 1.75\delta[n-3] + 0.75\delta[n-4] - 0.25\delta[n-5]$$

(c) Use multiplication of z-transform polynomials to find the output when the input is

$$x[n] = \delta[n-2] + 2\delta[n-4] - \delta[n-5].$$

First of all, we need to get $X(z)$, the z-transform of $x[n]$:

$$X(z) = z^{-2} + 2z^{-4}z^{-5}$$

Then we do the polynomial multiplication of $H(z)$ and $X(z)$:

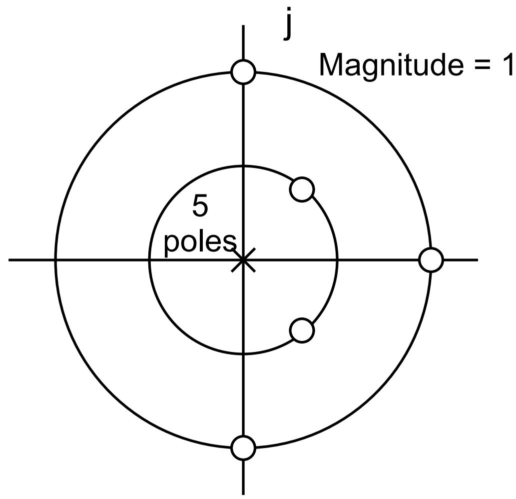
$$\begin{aligned} X(z)H(z) &= (z^{-2} + 2z^{-4}z^{-5})(1 - 1.5z^{-1} + 1.75z^{-2} - 1.75z^{-3} + 0.75z^{-4} - 0.25z^{-5}) \\ &= z^{-2} - 1.5z^{-3} + 3.75z^{-4} - 5.75z^{-5} + 5.75z^{-6} - 5.5z^{-7} + 3.25z^{-8} - 1.25z^{-9} + 0.25z^{-10} \end{aligned}$$

We can also do "synthetic multiplication" to form the following table of terms. We know from the exponents of z that the minimum delayed response is at $n = 2$, and the maximum delayed response is at $n = 10$.

Input	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$\delta[n-2]$	0	0	1	−1.5	1.75	−1.75	0.75	−0.25	0	0	0
$2\delta[n-4]$	0	0	0	0	2	−3	3.5	−3.5	1.5	−0.5	0
$-\delta[n-5]$	0	0	0	0	0	−1	1.5	−1.75	1.75	−0.75	0.25
Sum	0	0	1	−1.5	3.75	−5.75	5.75	−5.5	3.25	−1.25	0.25

$$\begin{aligned} y[n] = \delta[n-2] - 1.5\delta[n-3] + 3.75\delta[n-4] - 5.75\delta[n-5] + 5.75\delta[n-6] \\ -5.5\delta[n-7] + 3.25\delta[n-8] - 1.25\delta[n-9] + 0.25\delta[n-10] \end{aligned}$$

(d) Plot the poles and zeros of $H(z)$ in the z-plane.



(e) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.
We just replace z with $e^{j\omega}$, we get:

$$H(e^{j\hat{\omega}}) = 1 - 1.5e^{-j\hat{\omega}} + 1.75e^{-j2\hat{\omega}} - 1.75e^{-j3\hat{\omega}} + 0.75e^{-j4\hat{\omega}} - 0.25e^{-j5\hat{\omega}}$$

but it is not possible to do much simplification because the impulse response is not symmetric. Here is one way to simplify:

$$\begin{aligned}
H(e^{j\hat{\omega}}) &= (1 - e^{-j\hat{\omega}})(1 + e^{-j2\hat{\omega}})(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\
&= (1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}})(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\
&= \frac{1 - e^{-j4\hat{\omega}}}{1 - (-e^{-j\hat{\omega}})}(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\
&= e^{-j1.5\hat{\omega}} e^{j\pi/2} \left(\frac{\sin(2\hat{\omega})}{\cos(\hat{\omega}/2)} \right) (1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}})
\end{aligned}$$

(f) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of frequency $\hat{\omega}$ will the output signal be zero for all n (i.e., $y[n] = 0$) ? Find all possible frequencies in the range $-\pi \le \hat{\omega} \le \pi$. *Hint: Take a look at the locations of the zeros of $H(z)$ as plotted in part (d).*
We can use the factored form of $H(z)$, or the zeros of $H(z)$, to find the zeros that are *on the unit circle*. These are $z = 1$, $z = e^{j\pi/2}$ and $z = e^{-j\pi/2}$. These three zeros all have a magnitude equal to 1. Using the relationship between the z and $\hat{\omega}$ domains, $z = e^{j\hat{\omega}}$, we see that the three frequencies are $\hat{\omega} = 0$, $\hat{\omega} = \pi/2$ and $\hat{\omega} = -\pi/2$. The output for three different sinusoids will be zero. For example, when the frequency is $\hat{\omega} = \pi/2$

$$x[n] = e^{j(\pi/2)n} \quad \rightarrow \quad y[n] = H(e^{j\pi/2})e^{j(\pi/2)n} = 0$$