GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #8
SOLUTION

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PROBLEM 8.1*:

[prior text deleted]

Now here are some possible inputs. In each case, state which of the above (#1, #2, or #3) approaches you would use. There may not be a clear cut answer. Give the approach that you *think* will yield the solution with least effort. Outline your approach to solving the problem of finding the output of the 5-point moving averager.

(a) $x[n]$ is a sampled speech signal. It is represented by a vector of 10000 numbers.

Best approach: #1

Step 1 Find the difference equation for the filter.

Step 2 Compute $y[n]$ by iterating this difference equation (preferably using software, e.g., MATLAB) on the input data from $n = 0$ to $n = 9999$.

(b) $x[n] = u[n]$

Best approach given your knowledge going into this assignment: #1

Step 1 Find the difference equation for the filter.

Step 2 Compute $y[n]$ for $0 \leq n < 5$ by iterating this difference equation on the unit step function.

Step 3 Complete output by setting $y[n] = 0$ for $n < 0$, and $y[n] = 1$ for $n \geq 5$ (because the average of five ones is one).

Another approach given newly presented information(on IIR filters and infinite length signals) regarding $U(z)$: #2

Step 1 Find $U(z)$, the z-transform of $u[n]$, the unit step sequence.

Step 2 Find $H(z)$, the system function of the 5-point running averager.

Step 3 Multiply $U(z)H(z)$ to get $Y(z)$.

Step 4 Take the inverse z-transform of $Y(z)$ to get $y[n]$.

(c) $x[n] = 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi)$ for $-\infty < n < \infty$.

Best approach: #3

Step 1 Find $H(e^{j\hat{\omega}})$, the frequency response of the 5-point running averager.

Step 2 Evaluate the frequency response at the frequency of each of the cosine terms: $\hat{\omega} = 0.1\pi$ and $\hat{\omega} = 0.4\pi$.

Step 3 Find $y[n]$ by multiplying the amplitudes and adding the phases for each pair of cosine / frequency-response terms.

(d) $x[n] = \begin{cases} 1 & 0 \leq n \leq 10 \\ 0 & \text{otherwise.} \end{cases}$

Good approach: convolution

Step 1 Find $h[n]$, the unit impulse response of the 5-point running averager.

Step 2 Compute $y[n] = h[n] * x[n]$ ($x[n]$ was given).

(e) $x[n] = 10\delta[n-50]$

Good approach: convolution

Step 1 Find $h[n]$, the unit impulse response of the 5-point running averager.

Step 2 Compute $y[n] = h[n] * 10\delta[n-50]$, which gives an output that is a scaled and delayed version of $h[n]$, i.e, the output is $10h[n-50]$.

PROBLEM 8.2:

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) Find the output $y_1[n]$ when the input is

$$x_1[n] = 10\delta[n-50]$$

First, find $h[n]$, the finite impulse response of the filter:

$$h[n] = \frac{1}{5}\sum_{k=0}^{4}\delta[n-k]$$

Then convolve $h[n]$ with the input:

$$\begin{aligned} y_1[n] &= h[n] * 10\delta[n-50] \\ &= 10h[n-50] \\ &= 2\cdot\sum_{k=50}^{54}\delta[n-k] \end{aligned}$$

(b) Find the output $y_2[n]$ when the input is

$$x_2[n] = \begin{cases} 1 & 0 \le n \le 10 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the sequences for $x_2[n]$ and $h[n]$:

$$\begin{aligned} \{x_2[n]\} &= \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\} \\ \{h[n]\} &= \{1/5, 1/5, 1/5, 1/5, 1/5\} \end{aligned}$$

Convolve the two sequences:

$h[0] * x_2[n]$	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5				
$h[1] * x_2[n-1]$		1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5			
$h[2] * x_2[n-2]$			1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5		
$h[3] * x_2[n-3]$				1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	
$h[4] * x_2[n-4]$					1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5
$h[n] * x[n]$	1/5	2/5	3/5	4/5	1	1	1	1	1	1	1	4/5	3/5	2/5	1/5

The values of the output signal are given below, starting at $n = 0$ and ending at $n = 14$.

$$\{y_2[n]\} = \{1/5, 2/5, 3/5, 4/5, 1, 1, 1, 1, 1, 1, 1, 4/5, 3/5, 2/5, 1/5\}$$

(c) Find the output $y_3[n]$ when the input is

$$x_3[n] = 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty$$

Step 1 Find $H(e^{j\hat{\omega}})$, the frequency response of the 5-point running averager using the Dirichlet function:

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)} e^{-j2\hat{\omega}}$$

Step 2 Evaluate the frequency response at the frequency of each of the cosine terms.

$$\begin{aligned} H(e^{j0.1\pi}) &= \frac{\sin(0.5\pi/2)}{5\sin(0.1\pi/2)} e^{-j0.2\pi} = \frac{\sin(\pi/4)}{5\sin(\pi/20)} e^{-j\pi/5} \\ &\approx 0.904 e^{-j\pi/5} \end{aligned}$$

$$\begin{aligned} H(e^{j0.4\pi}) &= \frac{\sin(5(0.4\pi)/2)}{5\sin(0.4\pi/2)} e^{-j2(0.4\pi)} = \frac{\sin(\pi)}{5\sin(\pi/5)} e^{-j4\pi/5} \\ &= 0 \end{aligned}$$

Step 3 Find $y[n]$ by multiplying the amplitudes and adding the phases for each pair of cosine / frequency-response terms.

$$\begin{aligned} y_3[n] &\approx (4 \cdot 0.904)\cos(0.1\pi n + \pi/2 - \pi/5) + (3 \cdot 0)\cos(0.4\pi n - \pi) \\ &\approx 3.616\cos(0.1\pi n + 0.3\pi) \end{aligned}$$

(d) Use the concept of linearity to find the output $y_4[n]$ when the input is

$$x_4[n] = 10\delta[n-50] + 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty$$

From the previous parts, we see that $x_4[n] = x_1[n] + x_3[n]$. Thus, using linearity, we know that $y_4[n] = y_1[n] + y_3[n]$, resulting in

$$y_4[n] \approx 2 \cdot \sum_{k=50}^{54} \delta[n-k] + 3.616\cos(0.1\pi n + 0.3\pi)$$

The approximation sign is used because the value 3.616 is approximate.

PROBLEM 8.3:

A linear time-invariant system is described by the difference equation

$$y[n] = 0.8y[n-1] + x[n] + x[n-1].$$

(a) Suppose that the input is the unit step sequence, i.e., $x[n] = u[n]$. The output signal will be infinitely long. Determine by iterating the difference equation, the output of this system for the range $0 \le n \le 10$.

Recall that the unit step signal is zero for $n < 0$ and one for $n \ge 0$. The *initial rest condition* is our standard assumption for initial conditions when the IIR filter is to be a linear time-invariant system, so we have $y[n] = 0$ for $n < 0$, and then we can calculate the values one at time:

$$\begin{aligned}
y[0] &= (0.8)(0) + (1) + (0) = 1 \\
y[1] &= (0.8)(1) + (1) + (1) = 2.8 \\
y[2] &= (0.8)(2.8) + (1) + (1) = 4.24 \\
y[3] &= (0.8)(4.24) + (1) + (1) = 5.392 \\
y[4] &= (0.8)(5.392) + (1) + (1) \approx 6.314 \\
y[5] &\approx (0.8)(6.314) + (1) + (1) \approx 7.051 \\
y[6] &\approx (0.8)(7.051) + (1) + (1) \approx 7.641 \\
y[7] &\approx (0.8)(7.641) + (1) + (1) \approx 8.113 \\
y[8] &\approx (0.8)(8.113) + (1) + (1) \approx 8.490 \\
y[9] &\approx (0.8)(8.490) + (1) + (1) \approx 8.792 \\
y[10] &\approx (0.8)(8.792) + (1) + (1) \approx 9.034
\end{aligned}$$

• Notice that the general solution for $n \ge 0$ is $y[n] = 10 - 9(0.8)^n$, which contains a term that is “pole to the n.” Also $y[n]$ converges to 10 as $n \to \infty$.

(b) Suppose that the input is the pulse sequence

$$x[n] = u[n] - u[n-6] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

Once again, the output signal will be infinitely long. Determine the output of this system for $0 \le n \le 10$.

Iteration Method

If we define $y_b[n]$ to be the solution for this problem, and assume that $y_b[n] = 0$ for $n < 0$:

$$\begin{aligned}
y_b[0] &= (0.8)(0) + (1) + (0) = 1 \\
y_b[1] &= (0.8)(1) + (1) + (1) = 2.8 \\
y_b[2] &= (0.8)(2.8) + (1) + (1) = 4.24 \\
y_b[3] &= (0.8)(4.24) + (1) + (1) = 5.392 \\
y_b[4] &= (0.8)(5.392) + (1) + (1) \approx 6.314 \\
y_b[5] &\approx (0.8)(6.314) + (1) + (1) \approx 7.051 \\
y_b[6] &\approx (0.8)(7.051) + (0) + (1) \approx 6.641 \\
y_b[7] &\approx (0.8)(6.641) + (0) + (0) \approx 5.313 \\
y_b[8] &\approx (0.8)(5.313) + (0) + (0) \approx 4.250 \\
y_b[9] &\approx (0.8)(4.250) + (0) + (0) \approx 3.400 \\
y_b[10] &\approx (0.8)(3.400) + (0) + (0) \approx 2.720
\end{aligned}$$

- Notice that the general solution for $n \ge 6$ is $y_b[n] = 6.641(0.8)^{n-6}$, which is "pole to the n."

Applying Linearity and Time Invariance

If we define $y_b[n]$ to be the solution for this problem and use $y[n]$ from the previous part:

$$y_b[n] = y[n] - y[n-6]$$

resulting in the following solutions

$$\begin{aligned}
y_b[0] &= y[0] - y[-6] = 1 - 0 = 1 \\
y_b[1] &= y[1] - y[-5] = 2.8 - 0 = 2.8 \\
y_b[2] &= y[2] - y[-4] = 4.24 - 0 = 4.24 \\
y_b[3] &= y[3] - y[-3] = 5.392 - 0 = 5.392 \\
y_b[4] &= y[4] - y[-2] \approx 6.314 - 0 \approx 6.314 \\
y_b[5] &= y[5] - y[-1] \approx 7.051 - 0 \approx 7.051 \\
y_b[6] &= y[6] - y[0] \approx 7.641 - 1 \approx 6.641 \\
y_b[7] &= y[7] - y[1] \approx 8.113 - 2.8 \approx 5.313 \\
y_b[8] &= y[8] - y[2] \approx 8.490 - 4.24 \approx 4.250 \\
y_b[9] &= y[9] - y[3] \approx 8.792 - 5.392 \approx 3.400 \\
y_b[10] &= y[9] - y[3] \approx 9.034 - 6.314 \approx 2.720
\end{aligned}$$