GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #10
SOLUTION

Date: 5 Nov 99 (FRIDAY)

PROBLEM 10.1*:

Determine the z-transforms of the following sequences:

(a) $x_a[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

Use the fact that the transform of $\delta[n-k]$ is z^{-k} to derive

$$X_a(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

(b) $x_b[n] = u[n] - u[n-5]$

Use the fact that the transform of $u[n-k]$ is $z^{-k}/(1-z^{-1})$ to derive

$$X_b(z) = \frac{1}{1-z^{-1}} - \frac{z^{-5}}{1-z^{-1}} = \frac{1-z^{-5}}{1-z^{-1}}$$

Note: you can verify that $x_b[n] = x_a[n]$. Therefore, it must be true that $X_b(z) = X_a(z)$, even though the formulas look different.

(c) $x_c[n] = (0.8)^n u[n] + (-0.8)^n u[n]$.

Express your answer as: (1) a sum of rational functions; (2) a ratio of polynomials in z^{-1}; and (3) a product of factors of the form $(1 - az^{-1})$.

Use the fact that the transform of $a^n u[n]$ is $1/(1-az^{-1})$ to derive form (1):

$$X_c(z) = \frac{1}{1-0.8z^{-1}} + \frac{1}{1+0.8z^{-1}}$$

Then rearrange to get form (2):

$$\begin{aligned}
X_c(z) &= \frac{1+0.8z^{-1}}{(1-0.8z^{-1})(1+0.8z^{-1})} + \frac{1-0.8z^{-1}}{(1+0.8z^{-1})(1-0.8z^{-1})} \\
&= \frac{(1+0.8z^{-1}) + (1-0.8z^{-1})}{(1+0.8z^{-1})(1-0.8z^{-1})} \\
&= \frac{2}{(1+0.8z^{-1})(1-0.8z^{-1})} \\
&= \frac{2}{1-0.64z^{-2}}
\end{aligned}$$

Then rearrange to get form (3):

$$X_c(z) = 2 \cdot \frac{1}{1+0.8z^{-1}} \cdot \frac{1}{1-0.8z^{-1}}$$

(d) $x_d[n] = 2(0.8)^n \cos(0.5\pi n)u[n]$.

Express your answer as: (1) a sum of rational functions; (2) a ratio of polynomials in z^{-1}; and (3) a product of factors of the form $(1 - az^{-1})$.

Start by rearraging the sequence into two exponential terms:

$$\begin{aligned} x_d[n] &= 2(0.8)^n \cos(0.5\pi n)u[n] \\ &= 2(0.8)^n \frac{1}{2}\left(e^{j0.5\pi n} + e^{-j0.5\pi n}\right)u[n] \\ &= (0.8e^{j0.5\pi})^n u[n] + (0.8e^{-j0.5\pi})^n u[n] \\ &= (j0.8)^n u[n] + (-j0.8)^n u[n] \end{aligned}$$

then use the fact that the transform of $a^n u[n]$ is $1/(1 - az^{-1})$ to derive form (1):

$$X_d(z) = \frac{1}{1 - j0.8z^{-1}} + \frac{1}{1 + j0.8z^{-1}}$$

Then rearrange to get form (2):

$$\begin{aligned} X_d(z) &= \frac{1 + j0.8z^{-1}}{(1 - j0.8z^{-1})(1 + j0.8z^{-1})} + \frac{1 - j0.8z^{-1}}{(1 + j0.8z^{-1})(1 - j0.8z^{-1})} \\ &= \frac{(1 + j0.8z^{-1}) + (1 - j0.8z^{-1})}{(1 + j0.8z^{-1})(1 - j0.8z^{-1})} \\ &= \frac{2}{(1 + j0.8z^{-1})(1 - j0.8z^{-1})} \\ &= \frac{2}{1 + 0.64z^{-2}} \end{aligned}$$

Then rearrange to get form (3):

$$\begin{aligned} X_d(z) &= 2 \cdot \frac{1}{1 + j0.8z^{-1}} \cdot \frac{1}{1 - j0.8z^{-1}} \\ &= 2 \cdot \frac{1}{1 - 0.8e^{-j0.5\pi}z^{-1}} \cdot \frac{1}{1 - 0.8e^{j0.5\pi}z^{-1}} \end{aligned}$$

PROBLEM 10.2*:

Determine the inverse z-transforms of the following:

(a) $H_a(z) = 1 + 2z^{-2} + 4z^{-4} - 2z^{-6} - z^{-8}$

Use the fact that the inverse transform of z^{-k} is $\delta[n-k]$ to derive

$$h_a[n] = \delta[n] + 2\delta[n-2] + 4\delta[n-4] - 2\delta[n-6] - \delta[n-8]$$

(b) $H_b(z) = \dfrac{1+z^{-2}}{1-0.5z^{-1}}$

Start by breaking up $H_b(z)$ as follows:

$$H_b(z) = \frac{1}{1-0.5z^{-1}} + \frac{z^{-2}}{1-0.5z^{-1}}$$

then use the fact that the inverse transform of $z^{-k}/(1-az^{-1})$ is $a^{n-k}u[n-k]$ to derive

$$h_b[n] = (0.5)^n u[n] + (0.5)^{n-2} u[n-2] = \begin{cases} 0 & n < 0 \\ (0.5)^n & 0 \leq n < 2 \\ 5(0.5)^n & n \geq 2 \end{cases}$$

(c) $H_c(z) = \dfrac{2}{1-0.4z^{-1}} - \dfrac{1}{1+0.8z^{-1}}$

Use the fact that the inverse transform of $1/(1-az^{-1})$ is $a^n u[n]$ to derive

$$h_c[n] = 2(0.4)^n u[n] - (-0.8)^n u[n] = \left[2(0.4)^n - (-0.8)^n\right] u[n]$$

(d) *Note: This is the corrected form as described on the bulletin board (the original form follows).*

$$\begin{aligned} H_d(z) &= \frac{1+2z^{-1}}{1+0.4z^{-1}-0.32z^{-2}} \\ &= \frac{1+2z^{-1}}{(1-0.4z^{-1})(1+0.8z^{-1})} \end{aligned}$$

Next use partial fractions to derive the following form:

$$H_d(z) = \frac{A}{1-0.4z^{-1}} + \frac{B}{1+0.8z^{-1}}$$

which gives the following equation for that equates the numerators of the previous two equations:

$$A(1+0.8z^{-1}) + B(1-0.4z^{-1}) = 1 + 2z^{-1}$$

By isolating the z^0 and z^{-1} terms, we can derive

$$\begin{aligned} A + B &= 1 \\ 0.8A - 0.4B &= 2 \end{aligned}$$

resulting in $A = 2$ and $B = -1$ and the following equation:

$$H_d(z) = \frac{2}{1-0.4z^{-1}} - \frac{1}{1+0.8z^{-1}}$$

Thus, $H_d(z) = H_c(z)$ and

$$h_d[n] = h_c[n] = \left[2(0.4)^n - (-0.8)^n\right] u[n]$$

Note: This is the original form.

$$H_d(z) = \frac{1+2z^{-1}}{1-0.4z^{-1}+0.32z^{-2}}$$

The quadratic formula gives the following roots for the denominator:

$$\begin{aligned} z &= 0.2(1 \pm j\sqrt{7}) \\ &= re^{\pm j\hat{\omega}_0} \end{aligned}$$

where $r \approx 0.5657$ and $\hat{\omega}_0 \approx 1.209$, and resulting in the following equation:

$$H_d(z) = \frac{1+2z^{-1}}{(1-re^{j\hat{\omega}_0}z^{-1})(1-re^{-j\hat{\omega}_0}z^{-1})}$$

Next use partial fractions to derive the following form:

$$H_d(z) = \frac{A}{1-re^{j\hat{\omega}_0}z^{-1}} + \frac{B}{1-re^{-j\hat{\omega}_0}z^{-1}}$$

which gives the following equation for that equates the numerators of the previous two equations:

$$A(1-re^{-j\hat{\omega}_0}z^{-1}) + B(1-re^{j\hat{\omega}_0}z^{-1}) = 1+2z^{-1}$$

By isolating the z^0 and z^{-1} terms, we can derive

$$\begin{aligned} A+B &= 1 \\ -Are^{-j\hat{\omega}_0} - Bre^{j\hat{\omega}_0} &= 2 \end{aligned}$$

resulting in $A = be^{-j\phi}$ and $B = be^{j\phi}$ and the following equation:

$$H_d(z) = \frac{be^{-j\phi}}{1-re^{j\hat{\omega}_0}z^{-1}} + \frac{be^{j\phi}}{1-re^{-j\hat{\omega}_0}z^{-1}}$$

where $b \approx 2.138$ and $\phi \approx 1.335$.

Use the fact that the inverse transform of $1/(1-az^{-1})$ is $a^n u[n]$ to derive

$$\begin{aligned} h_d[n] &= be^{-j\phi}r^n e^{j\hat{\omega}_0 n}u[n] + be^{j\phi}r^n e^{-j\hat{\omega}_0 n}u[n] \\ &= br^n\left(e^{-j\phi}e^{j\hat{\omega}_0 n} + e^{j\phi}e^{-j\hat{\omega}_0 n}\right)u[n] \\ &= 2br^n\cos(\hat{\omega}_0 n - \phi)u[n] \end{aligned}$$

where the approximate values of a, b, $\hat{\omega}_0$, and ϕ are given above.

PROBLEM 10.3*:
A linear time-invariant filter is described by the difference equation

$$y[n] = 0.8y[n-1] - 0.8x[n] + x[n-1]$$

(a) Determine the system function $H(z)$ for this system. Express $H(z)$ as a ratio of polynomials in z^{-1} and as a ratio of polynomials in z.

Take the Z transform of the difference equation:

$$Y(z) = 0.8z^{-1}Y(z) - 0.8X(z) + z^{-1}X(z)$$

and rearrange to get the following ratio of polynomials in z^{-1}:

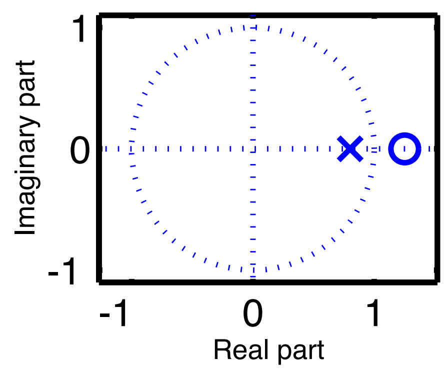
$$H(z) = \frac{Y(z)}{X(z)} = \frac{-0.8 + z^{-1}}{1 - 0.8z^{-1}} = -0.8 \cdot \frac{1 - 1.25z^{-1}}{1 - 0.8z^{-1}}$$

Multiply both numerator and denominator by z to derive

$$H(z) = -0.8 \cdot \frac{z - 1.25}{z - 0.8}$$

(b) Plot the poles and zeros of $H(z)$ in the z-plane.

There is a pole at $z = 0.8$ and a zero at $z = 1.25$.



(c) From $H(z)$, obtain an expression for $H(e^{j\hat{\omega}})$, the frequency response of the system.

$$H(e^{j\hat{\omega}}) = \frac{-0.8 + e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

(d) Show that $|H(e^{j\hat{\omega}})|^2 = 1$ for all $\hat{\omega}$.

$$\begin{aligned}
|H(e^{j\hat{\omega}})|^2 &= H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) \\
&= \frac{-0.8 + e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \cdot \frac{-0.8 + e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}} \\
&= \frac{0.64 - 0.8e^{j\hat{\omega}} - 0.8e^{-j\hat{\omega}} + 1}{1 - 0.8e^{j\hat{\omega}} - 0.8e^{-j\hat{\omega}} + 0.64} \\
&= 1
\end{aligned}$$

(e) If the input to the system is

$$x[n] = 4 + \cos[(\pi/4)n] - 3\cos[(2\pi/3)n]$$

what can you say, without further calculation, about the form of the output $y[n]$?

Because $|H(e^{j\hat{\omega}})|^2 = 1$, we know that the amplitude of each of the sinusoids (and the value of the constant term) in $y[n]$ is the same as it is in $x[n]$. Thus, we get

$$y[n] = 4 + \cos[(\pi/4)n + \phi_1] - 3\cos[(2\pi/3)n + \phi_2]$$

but the phases ϕ_1 and ϕ_2 are harder to determine. The phase at zero frequency will be zero.

PROBLEM 10.4*:

Match the frequency responses (A–E) with the correct pole–zero plots (PZ 1–6).

(a) PZ2: A pole and zero cancel at $\hat{\omega} = \pi$, giving a large peak at that frequency.

(b) PZ5: The pole at $z \approx -1$ makes the response very large at $\hat{\omega} \approx \pi$. There is also a zero at $\hat{\omega} = 0$.

(c) PZ1: A pole and zero cancel at $\hat{\omega} = 0$, giving a large peak at that frequency.

(d) PZ6: The poles at $z \approx \pm j$ makes the response very large at $\hat{\omega} = \pm\pi/2$. There is also a zero at $\hat{\omega} = 0$.

(e) PZ3: There are peaks for each of the five poles (except that there isn't an explicit peak at $\hat{\omega} \approx \pi$). There is also a zero at $\hat{\omega} = 0$.

PROBLEM 10.5*:

An LTI system has the following system function:

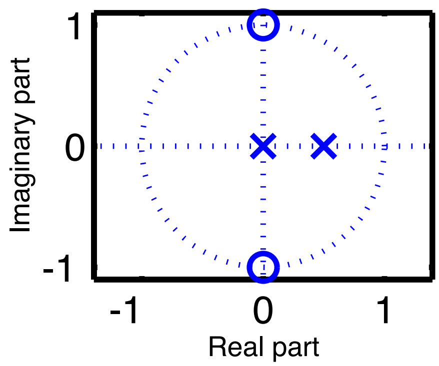
$$H(z) = \frac{1 + z^{-2}}{1 - 0.5z^{-1}}$$

(a) Plot the poles and zeros of $H(z)$ in the z-plane.

Multiply the numerator and denominator by z^2 to get

$$H(z) = \frac{z^2 + 1}{z^2 - 0.5z} = \frac{(z - j)(z + j)}{z(z - 0.5)}$$

Thus, there are poles at $z = 0$ and $z = 0.5$ and zeros at $z = j$ and $z = -j$.



(b) Determine the difference equation that is satisfied by the general input $x[n]$ and the corresponding output $y[n]$ of the system.

The denominator of $H(z)$ gives the terms of $y[n]$ and the numerator gives the terms of $x[n]$, resulting in

$$y[n] = 0.5y[n-1] + x[n] + x[n-2]$$

(c) Use z-transforms to determine the impulse response $h[n]$ of the system; i.e., the output of the system when the input is $x[n] = \delta[n]$.

Start by rearranging the system function as

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{z^{-2}}{1 - 0.5z^{-1}}$$

Then use the fact that the inverse transform of $z^{-k}/(1 - az^{-1})$ is $a^{n-k}u[n-k]$ to derive

$$h[n] = (0.5)^n u[n] + (0.5)^{n-2} u[n-2] = \begin{cases} 0 & n < 0 \\ (0.5)^n & 0 \le n < 2 \\ 5(0.5)^n & n \ge 2 \end{cases}$$

(d) Determine an expression for the frequency response $H(e^{j\hat{\omega}})$ of the system.

$$H(e^{j\hat{\omega}}) = \frac{1 + e^{-j2\hat{\omega}}}{1 - 0.5e^{-j\hat{\omega}}}$$

(e) Use the frequency response function to determine the output $y_1[n]$ of the system when the input is

$$x_1[n] = 2\cos(0.5\pi n) \qquad -\infty < n < \infty.$$

First determine the frequency response at $\hat{\omega} = 0.5\pi$ (the frequency of the sinusoidal term):

$$\begin{aligned} H(e^{j\pi/2}) &= \frac{1+e^{-j\pi}}{1-0.5e^{-j\pi/2}} \\ &= \frac{1+(-1)}{1-0.5(-j)} \\ &= 0 \end{aligned}$$

Because the frequency response at $\hat{\omega} = 0.5\pi$ is equal to 0, $y_1[n] = 0$ for $-\infty < n < \infty$.

(f) Use the z-transform to determine the output $y_2[n]$ when the input is

$$x_2[n] = 2\cos(0.5\pi n)u[n] = \begin{cases} 2\cos(0.5\pi n) & n \geq 0 \\ 0 & n < 0. \end{cases}$$

First rewrite $x_2[n]$ as follows:

$$x_2[n] = e^{j0.5\pi n}u[n] + e^{-j0.5\pi n}u[n]$$

Then, use the fact that the transform of $a^n u[n]$ is $1/(1-az^{-1})$ to derive

$$\begin{aligned} X_2(z) &= \frac{1}{1-e^{j0.5\pi}z^{-1}} + \frac{1}{1-e^{-j0.5\pi}z^{-1}} \\ &= \frac{1}{1-jz^{-1}} + \frac{1}{1+jz^{-1}} \\ &= \frac{1+jz^{-1}}{(1-jz^{-1})(1+jz^{-1})} + \frac{1-jz^{-1}}{(1-jz^{-1})(1+jz^{-1})} \\ &= \frac{2}{1+z^{-2}} \end{aligned}$$

Compute the transform of the output by

$$\begin{aligned} Y_2(z) &= X_2(z)H(z) \\ &= \left(\frac{2}{1+z^{-2}}\right)\left(\frac{1+z^{-2}}{1-0.5z^{-1}}\right) \\ &= \frac{2}{1-0.5z^{-1}} \end{aligned}$$

Finally, use the fact that the inverse transform of $1/(1-az^{-1})$ is $a^n u[n]$ to derive

$$y_2[n] = 2(0.5)^n u[n]$$

Note that in this case, only the "transient response" is non-zero. We showed in part(e) that the steady-state response is exactly zero.