GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #11

Assigned: 5 November
Due Date: 12 November 99 (FRIDAY)

PROBLEM 11.1*:

A linear time-invariant system has system function

$$H(z) = \frac{0.0795(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3}} \quad (1)$$

$$= \frac{0.0795(1+z^{-1})(1-e^{j0.5\pi}z^{-1})(1-e^{-j0.5\pi}z^{-1})}{(1-0.556z^{-1})(1-0.846e^{j0.3\pi}z^{-1})(1-0.846e^{-j0.3\pi}z^{-1})} \quad (2)$$

$$= -0.2 + \frac{A}{1-0.556z^{-1}} + \frac{0.17e^{j0.96\pi}}{1-0.846e^{j0.3\pi}z^{-1}} + \frac{0.17e^{-j0.96\pi}}{1-0.846e^{-j0.3\pi}z^{-1}} \quad (3)$$

(a) Use equation (1) to determine the difference equation that relates the input $x[n]$ to the output $y[n]$ for this system.

Solution is to use (1) and copy the filter coefficients from $H(z)$ into a difference equation:

$$y[n]-1.556y[n-1]+1.272y[n-2]-0.398y[n-3] = 0.0795(x[n]+x[n-1]+x[n-2]+x[n-3]]$$

And it is usually the convention to put $y[n]$ by itself on the left-hand side of the equation:

$$y[n] = 1.556y[n-1]-1.272y[n-2]+0.398y[n-3]+0.0795(x[n]+x[n-1]+x[n-2]+x[n-3]]$$

(b) Starting with equation(2), use the partial fraction expansion method to show that $A = 0.62$ in equation (3).

Solution: Multiply (2) and (3) by $(1-0.556z^{-1})$. In (2), this just cancels the factor $(1-0.556z^{-1})$. In (3), this puts the factor in the numerator of every term except the one with A. Since this expansion must be valid for all values of z, we evaluate these functions at $z = 0.556$. Subsituting this value into MATLAB, we get the result that $A = 0.6172$. We used the following MATLAB commands:

```
aa = 1/0.556
A_numerator = 0.0795*(1+aa)*(1-j*aa)*(1+j*aa)
A_denominator = (1-0.846*exp(j*.3*pi)*aa)*(1-0.846*exp(-j*.3*pi)*aa)
A = A_numerator / A_denominator
```

Another way to do it is to think of the operations needed to put (3) over a common denominator. When you do that, you would cross-multiply by the denominator factors. The result in the numerator would be that the constant term of the numerator is:

$$-0.2 + A + 0.17e^{j0.96\pi} + 0.17e^{-j0.96\pi}$$

which according to (1) would have to be equal to 0.0795. Thus we get

$$A = 0.0795 + 0.2 - 0.17e^{j0.96\pi} - 0.17e^{-j0.96\pi} = 0.2795 - 0.34\cos(0.96\pi) \approx 0.6172$$

(c) Determine the impulse response of this system by finding the inverse z-transform of equation (3). Express your answer in the form

$$h[n] = A_0\delta[n] + A_1 a^n u[n] + A_2 r^n \cos(\hat{\omega}_0 n + \phi).$$

Solution: Recall that

$$H_1(z) = \frac{1}{1-az^{-1}} \rightarrow h_1[n] = a^n u[n]$$

Therefore, we get the impulse response as

$$h[n] = -0.2\delta[n] + 0.6172(0.556)^n u[n] + 0.17e^{j0.96\pi}(0.846)^n e^{j0.3\pi n}u[n] + 0.17e^{-j0.96\pi}(0.846)^n e^{-j0.3\pi n}u[n]$$

which can be simplified to:

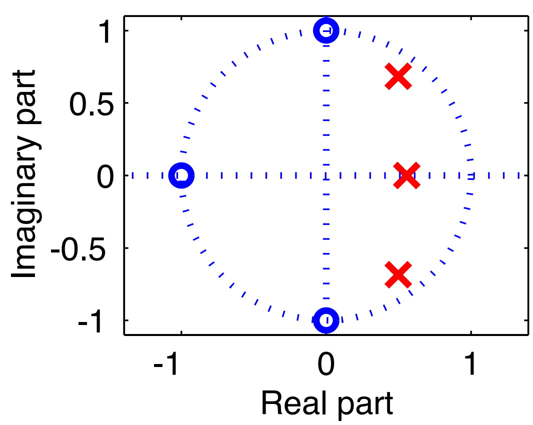
$$h[n] = -0.2\delta[n] + 0.6172(0.556)^n u[n] + 0.34(0.846)^n \cos(0.3\pi n + 0.96\pi)u[n]$$

(d) Use equation (2) to plot the pole and zero locations for this system.

Solution: We have poles at $z = 0.556$, $z = 0.846e^{j0.3\pi}$, and $z = 0.846e^{-j0.3\pi}$.

We have zeros at $z = 1,\ j, -j$.

We show the plot of these poles and zeros below. We can use
`zplane([-1,j,-j].',[0.556*exp(j*0) 0.846*exp(j*0.3*pi) 0.846*exp(-0.3*j*pi)].')`
to make the plot.



(e) The frequency response of the system can be computed in MATLAB using the following statements:

```
omegahat = -pi:(pi/200):pi;
aa =
bb =
HH = freqz(bb,aa,omegahat);
```

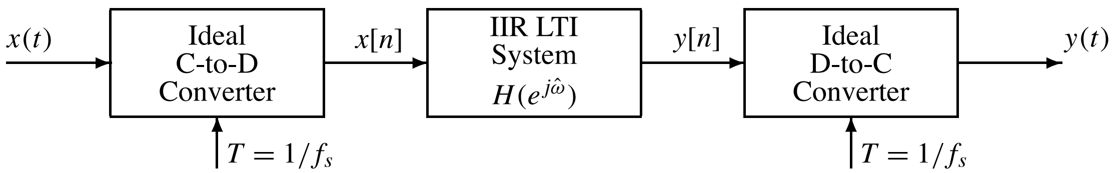
Determine the coefficient vectors aa and bb.

Solution: The MATLAB statements are:

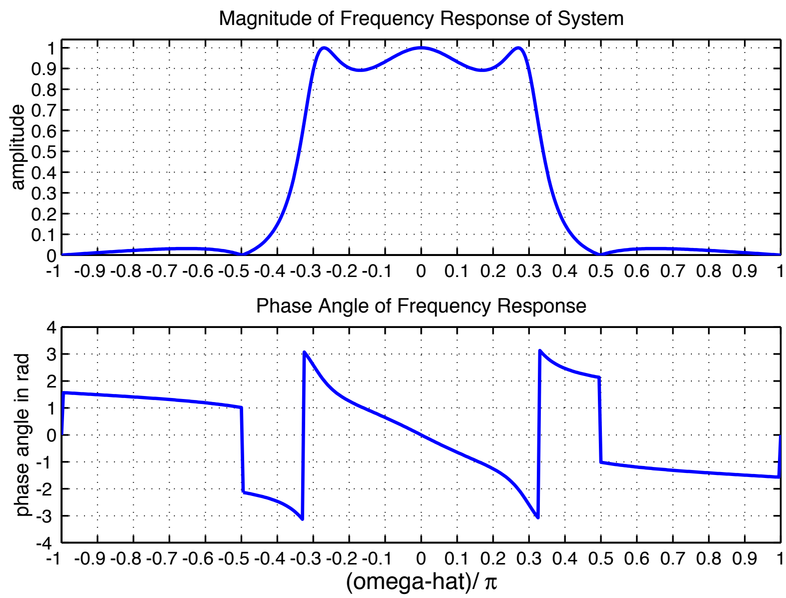
```
b = 0.0795*[1 1 1 1];
a = [1 -1.556 1.272 -0.398];
```

PROBLEM 11.2*:

Consider the following system for filtering continuous-time signals using an IIR digital filter.



The frequency response of the IIR system, which is the IIR system of Problem 11.1, is plotted in the following figure:



If the sampling rate is $f_s = 10000$ samples/second, and the input to the C-to-D converter is

$$x(t) = 10 + 10\cos(2000\pi t) + 10\cos(5000\pi t - 2\pi/3),$$

use the above frequency response plots to determine an expression for $y(t)$, the output of the D-to-C converter.

Solution: Our sampled signal is represented by

$$x[n] = 10 + 10\cos(0.2\pi n) + 10\cos(0.5\pi n - 2\pi/3)$$

because we convert from analog frequency in rea/sec to digital frequency by dividing by the sampling frequency. Also the is no aliasing to worry about. To find the output response, we need to compute the frequency response at all three frequencies—we can read these values off of the plots.

At $\hat{\omega} = 0.5\pi$, the amplitude is 0. Also, the phase jump of π tells us that we went through a zero.

At $\hat{\omega} = 0$, the amplitude is 1, with zero phase shift.

At $\hat{\omega} = 0.2\pi$, the amplitude is approximately 0.9, and the phase is roughly 1.4 radians.

Therefore, the discrete output is

$$\begin{aligned} y[n] &= 10(1) + 10(0.9)\cos(0.2\pi n + 1.4) + 10(0)\cos(0.5\pi n - 2\pi/3) \\ y[n] &= 10 + 9\cos(0.2\pi n + 1.4) \end{aligned}$$

The output, assuming an ideal discrete to continuous converter is

$$y(t) = 10 + 9\cos(2000\pi t + 1.4)$$

PROBLEM 11.3*:

A continuous-time system is defined by the input/output relation

$$y(t) = \int_{t-2}^{t-1} x(\tau)d\tau.$$

(a) Show that this is a linear time-invariant system.

Solution: First, we test linearity. If we input a signal $x(t) = a_1x_1(t) + a_2x_2(t)$, then the output response is

$$\begin{aligned} y(t) &= \int_{t-2}^{t-1} (a_1x_1(\tau) + a_2x_2(\tau))d\tau \\ &= a_1\left(\int_{t-2}^{t-1} x_1(\tau)d\tau\right) + a_2\left(\int_{t-2}^{t-1} x_2(\tau)d\tau\right) \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

where $y_1(t)$, $y_2(t)$ are the responses to $x_1(t)$, $x_2(t)$.

In order to test time-invariance, we input a signal $x(t) = x_2(t - t_\text{d})$, then the output response is

$$\begin{aligned} y(t) &= \int_{t-2}^{t-1} x_2(\tau - t_d)d\tau \\ &= \int_{t-2}^{t-1} x_2(\tau - t_d)d\tau \\ &= \int_{t-2-t_d}^{t-1-t_d} x_2(\lambda)d\lambda \\ &= \int_{(t-t_d)-2}^{(t-t_d)-1} x_2(\lambda)d\lambda \;=\; y_2(t - t_d) \end{aligned}$$

where we replaced $(\tau - t_d)$ with λ, and $d\tau$ with $d\lambda$ in step #3.

(b) Determine the impulse response, $h(t)$, of this system. Plot it.

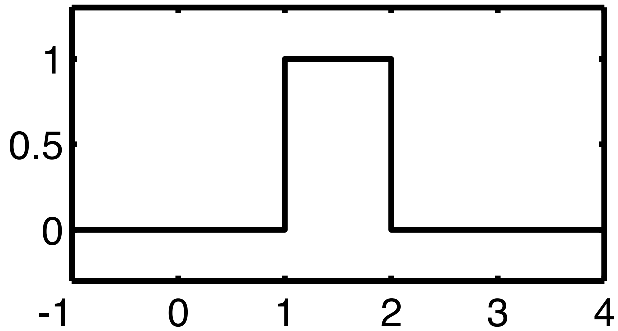
Solution:

$$\begin{aligned} h(t) &= y(t) = \int_{t-2}^{t-1} x(\tau)d\tau \\ &= y(t) = \int_{t-2}^{t-1} \delta(\tau)d\tau \end{aligned}$$

The integral of $\delta(\tau)$ will be non-zero if the limits of integration include the location of the impulse at $\tau = 0$. This is true if $(t-2) < 0$ and $(t-1) > 0$, or equivalently, the impulse response is equal to one when $1 < t < 2$; it is zero everywhere else. Therefore, the impulse response can be written in the following clever fashion.

$$h(t) = u(t-1) - u(t-2)$$

The difference of the two unit-step signals is actually a pulse of length one. We show the plot below.



(c) Is this a stable system? Is it a causal system?

Solution: This system is stable since the total integral of $|h(t)|$ is finite.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{1}^{2} |h(t)|dt = 1 < \infty$$

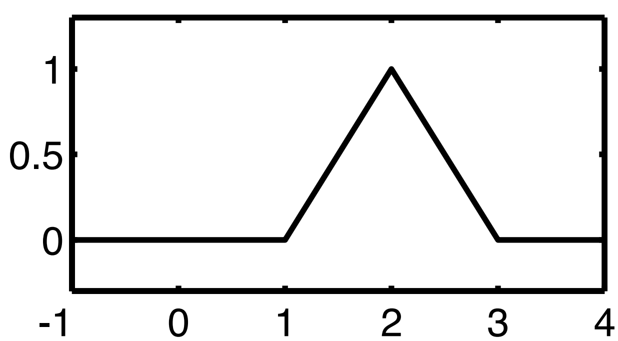
The system is causal because the impulse response does not start before $t = 0$. Thus the output signal $y(t)$ at time t only requires values of the input in the time reange $t-1$ to $t-2$ which is prior to time t.

(d) Use convolution to determine the output of the system when the input is

$$x(t) = u(t) - u(t-1) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Plot your answer.

Solution: This problem is the convolution of two boxes, of unit height and unit width. The solution is a triangle function of width 2 with its peak value equal to 1. In this case, since the box for $h(t)$ is delayed one second in time, the solution is delayed by one second in time. We plot this solution below.

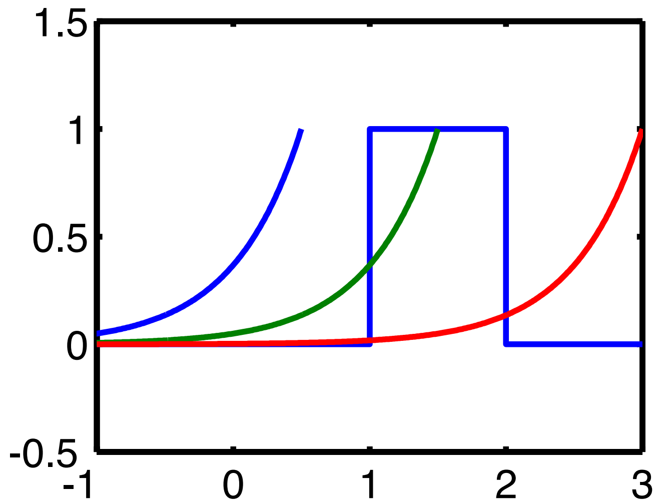


(e) Use convolution to determine the output when the input is

$$x(t) = e^{-2t}u(t).$$

Solution: We define convolution as

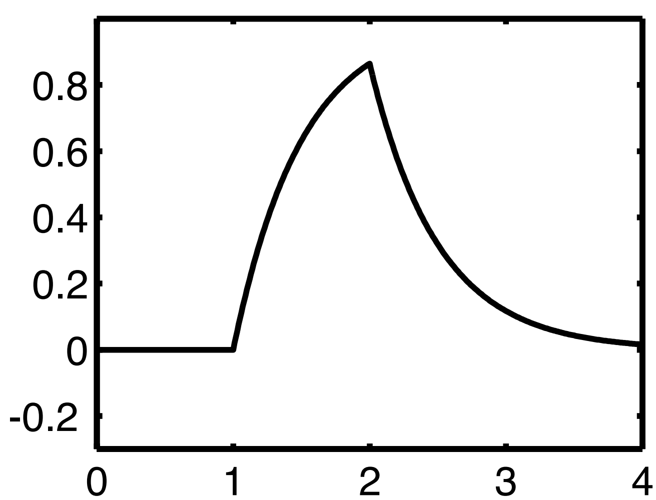
$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



We have three cases, illustrated by the figure above. We choose to flip the input, and not $h(t)$. In the first case, $t < 1$, there is no overlap between between $h(\tau)$ and $x(\tau - t)$. In the second and third cases, $1 < t$, there is overlap. In the third case, the overlap ends at $\tau = 2$. We generated the above plot using the following MATLAB commands:

```
xx1 = [0:0.01:4];
subplot(2,2,1)
plot([-1 1 1 2 2 3],[0 0 1 1 0 0],'b-')
hold on
plot(xx1-3.5,exp(2*(xx1 - 4)),xx1-2.5,exp(2*(xx1 - 4)),xx1-1,exp(2*(xx1 - 4)));
hold off
axis([-1 3 -0.5 1.5]);
```

The solution is $y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{2}\left(1 - e^{-2(t-1)}\right) & 1 < t < 2 \\ \frac{1}{2}e^{-2(t-2)}\left(1 - e^{-2}\right) & t > 2 \end{cases}$



We generated this plot with the following MATLAB commands:

```
xx = [0:0.01:1];
xx2 = [0:0.01:2];
plot([0 xx+1 xx2+2],[0 (1 - exp(-2*xx)) exp(-2*xx2)*(1 - exp(-2) )]);
```

Here is the integral for case #2 which applies when $1 < t < 2$:

$$\begin{aligned} \int_1^t e^{-2(t-\tau)} d\tau &= \tfrac{1}{2} e^{-2t}\, e^{2\tau}\Big|_1^t \\ &= \tfrac{1}{2} e^{-2t}\left(e^{2t} - 1\right) = \tfrac{1}{2}\left(1 - e^{-2(t-1)}\right) \end{aligned}$$

Here is the integral for case #3 which applies when $2 < t$:

$$\begin{aligned} \int_1^2 e^{-2(t-\tau)} d\tau &= \tfrac{1}{2} e^{-2t}\, e^{2\tau}\Big|_1^2 \\ &= \tfrac{1}{2} e^{-2t}\left(e^{4} - e^{2}\right) = \tfrac{1}{2}\left(e^{-2(t-2)} - e^{-2(t-1)}\right) \end{aligned}$$

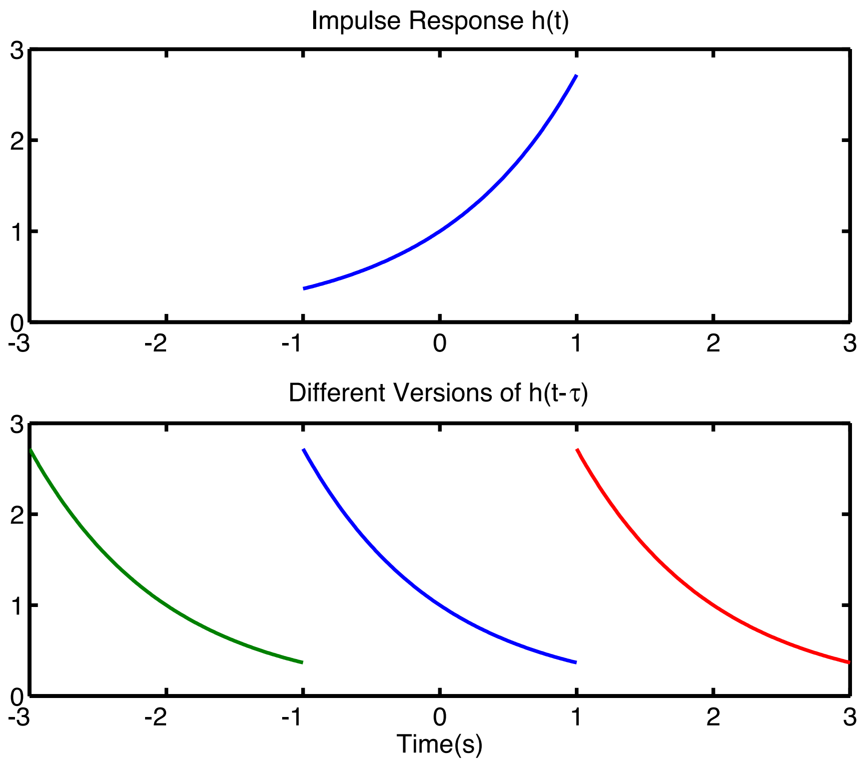
PROBLEM 11.4*:
A linear time-invariant system has impulse response:

$$h(t) = \begin{cases} e^t & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot $h(\tau)$ and $h(t-\tau)$ as a functions of τ for $t = -2, 0,$and 2.

Solution: We show the plot below. The MATLAB commands to generate this plot were:

```
xx4 = [-1:0.01:1];
subplot(2,1,1),plot(xx4,exp(xx4));
title('Impulse Response h(t)');
axis([-3 3 0 3])
subplot(2,1,2),plot(xx4,exp(-xx4),xx4-2,exp(-xx4),xx4+2,exp(-xx4));
xlabel('Time(s)');title('Different Versions of h(t-\tau)');
```



(b) Is the system stable? Justify your answer.

Solution: The system is stable, because the integral of $|h(t)|$ over all t is finite.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-1}^{1} |e^t|dt = e - e^{-1} < \infty$$

(c) Is the system causal? Justify your answer.

Solution: The system is not causal, because the impulse response $h(t)$ is non-zero before $t = 0$.

(d) Find the output $y(t)$ when the input is $x(t) = \delta(t-2)$.

Answer:

The delta function delays the impulse response by 2. We call this the shifting property of convolving with impulses:

$$\delta(t-2) * h(t) = h(t-2)$$

Therefore the solution is

$$h(t) = \begin{cases} e^{t-2} & 1 < t < 3 \\ 0 & \text{otherwise} \end{cases}$$

Notice that we must also shift the starting and ending limits on $h(t)$ by 2 secs.

(e) Use convolution to determine the output $y(t)$ when the input is $x(t) = \begin{cases} e^{-t} & 0 < t < 3 \\ 0 & \text{otherwise.} \end{cases}$

Answer: There are FIVE cases:

Case #1: For $t < -1$, there is no overlap between $x(t-\tau)$ and $h(\tau)$. Thus the output is zero.

Case #2: For $-1 < t < 1$, we must do the following integral:

$$\begin{aligned} \int_{-1}^{t} e^{\tau} e^{-(t-\tau)} d\tau &= \tfrac{1}{2} e^{-t} \, e^{2\tau} \Big|_{-1}^{t} \\ &= \tfrac{1}{2} e^{-t} \left(e^{2t} - e^{-2} \right) \; = \; \tfrac{1}{2} \left(e^{t} - e^{-(t+2)} \right) \end{aligned}$$

Case #3: For $1 < t < 2$, we must do the following integral:

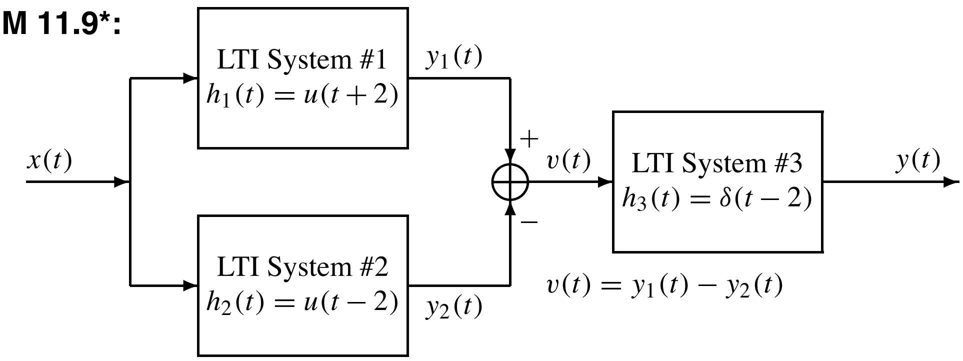
$$\begin{aligned} \int_{-1}^{1} e^{\tau} e^{-(t-\tau)} d\tau &= \tfrac{1}{2} e^{-t} \, e^{2\tau} \Big|_{-1}^{1} \\ &= \tfrac{1}{2} e^{-t} \left(e^{2} - e^{-2} \right) \; = \; \tfrac{1}{2} \left(e^{-(t-2)} - e^{-(t+2)} \right) \end{aligned}$$

Case #4: For $2 < t < 4$, we must do the following integral:

$$\begin{aligned} \int_{t-3}^{1} e^{\tau} e^{-(t-\tau)} d\tau &= \tfrac{1}{2} e^{-t} \, e^{2\tau} \Big|_{t-3}^{1} \\ &= \tfrac{1}{2} e^{-t} \left(e^{2} - e^{2(t-3)} \right) \; = \; \tfrac{1}{2} \left(e^{-(t-2)} - e^{t-6} \right) \end{aligned}$$

Case #5: For $t > 4$, there is no overlap; therefore the solution is equal to zero in this region.

PROBLEM 11.9*:



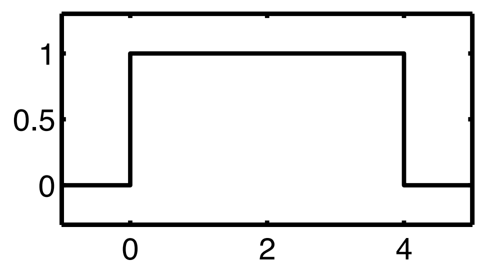
1. What is the impulse response of the overall LTI system (i.e., from $x(t)$ to $y(t)$)? Give your answer both as an equation and as a carefully labeled sketch.

   Solution:

$$\begin{aligned} v(t) &= h_1(t) * x(t) - h_2(t) * x(t) = (h_1(t) + h_2(t)) * x(t) \\ &= (u(t+2) - u(t-2)) * x(t) \end{aligned}$$

$$\begin{aligned} y(t) &= \delta(t-2) * v(t) = \delta(t-2) * (u(t+2) - u(t-2)) * x(t) \\ &= (u(t) - u(t-4)) * x(t) = h(t) * x(t) \end{aligned}$$

   Therefore, $h(t) = u(t) - u(t-4)$



   MATLAB code to generate this plot: `plot([-1 0 0 4 4 5],[0 0 1 1 0 0]);`

2. Is the overall system a causal system? (Explain to receive credit.) Is it a stable system? (Explain to receive credit.)

   Solution: This solution is causal, since the impulse response, $h(t) = u(t) - u(t-4)$, is zero for $t < 0$. This solution is stable, since the integral over $|h(t)|$ for all t is finite:

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^4 1\, dt = 4 < \infty$$

3. Are all the subsystems causal? (Explain to receive credit.) Are all the subsystems stable? (Explain to receive credit.)

   Solution:

   $h_1(t)$ is not causal, because its impulse response starts before t=0.

   $h_1(t)$ and $h_2(t)$ are not stable because the integral over all time of a unit-step signal is not finite. For example,

$$\int_{-\infty}^{\infty} |h_1(t)| dt = \int_{-2}^{\infty} 1\, dt \to \infty$$