GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #13

Assigned: 23 November 99
Due Date: 3 December 99 (FRIDAY)

PROBLEM 13.1*:

(a) $x(t) = 2\delta(t-2) \quad \rightarrow \quad X(j\omega) = 2e^{-2j\omega}$ (Delay property)

(b) $x(t) = 10 \quad \rightarrow \quad X(j\omega) = 20\pi\delta(\omega)$ (Fourier transform of impulses)

(c) $x(t) = \dfrac{\sin 20\pi(t-1)]}{\pi(t-1)} \quad \rightarrow \quad X(j\omega) = (u(\omega+20\pi) - u(\omega-20\pi))e^{-j\omega}$ (Delay property)

(d) $x(t) = \dfrac{2}{\pi}\cos(0.1\pi t) \quad \rightarrow \quad X(j\omega) = 2\delta(\omega-0.1\pi) + 2\delta(\omega+0.1\pi)$

(e) $x(t) = \dfrac{d}{dt}\left[\dfrac{\sin(20\pi t)]}{\pi t}\right] \quad \rightarrow \quad X(j\omega) = j\omega(u(\omega+20\pi) - u(\omega-20\pi))$ (Derivatives of signals)

(f) $x(t) = \dfrac{\sin(400\pi t)}{\pi t}\cos(20000\pi t) \quad \rightarrow$

$X(j\omega) = \dfrac{1}{2\pi}(u(\omega+400\pi) - u(\omega-400\pi)) * \pi(\delta(\omega-20000\pi) + \delta(\omega+20000\pi))$

$= \frac{1}{2}(u(\omega-19600\pi) - u(\omega-20400\pi)) + \frac{1}{2}(u(\omega+20400\pi) - u(\omega+19600\pi))$

(The * means convolution in the frequency domain.)

Or we can use the frequency shifting property of the cosine as two complex exponentials:

$x(t) = \dfrac{\sin(400\pi t)}{\pi t}\frac{1}{2}e^{j20000\pi t} + \dfrac{\sin(400\pi t)}{\pi t}\frac{1}{2}e^{-j20000\pi t} \quad \rightarrow$

$X(j\omega) = \frac{1}{2}(u(\omega-19600\pi) - u(\omega-20400\pi)) + \frac{1}{2}(u(\omega+20400\pi) - u(\omega+19600\pi))$

PROBLEM 13.2*:

(a) For the input signal,

$$x(t) = 10 + 2\delta(t-2) + \frac{\sin(2000\pi t)}{\pi t},$$

the Fourier transform can be done one term at a time:

$$X(j\omega) = 20\pi\delta(\omega) + 2e^{-j2\omega} + u(\omega + 2000\pi) - u(\omega - 2000\pi)$$

Note that the ideal LPF in frequency is represented by

$$u(\omega + 2000\pi) - u(\omega - 2000\pi) = \begin{cases} 1 & \text{for } |\omega| < 2000\pi \\ 0 & \text{for } |\omega| > 2000\pi \end{cases}$$

(b) We get the Fourier transform of the output by passing each term thru $H(j\omega)$ which has a cutoff frequency at $\omega_c = 1000\pi$.

$$\begin{aligned} Y_1(j\omega) &= 20\pi\delta(\omega)H(j\omega) = 20\pi\delta(\omega)H(j0) = 200\pi\delta(\omega) \\ Y_2(j\omega) &= 2e^{-j2\omega}H(j\omega) = 2e^{-j2\omega}[10u(\omega+1000\pi) - 10u(\omega-1000\pi)] \\ Y_3(j\omega) &= [u(\omega+2000\pi) - u(\omega-2000\pi)]\,H(j\omega) \\ &= [u(\omega+2000\pi) - u(\omega-2000\pi)]\,[10u(\omega+1000\pi) - 10u(\omega-1000\pi)] \\ &= 10[u(\omega+1000\pi) - u(\omega-1000\pi)] \end{aligned}$$

Adding these together gives the final result:

$$Y(j\omega) = 200\pi\delta(\omega) + (20e^{-j2\omega} + 10)[u(\omega+1000\pi) - u(\omega-1000\pi)]$$

(c) We can get y(t) by the inverse Fourier transform of the 3 terms in the equation above.

$$y(t) = 100 + 20\frac{\sin(1000\pi(t-2))}{\pi(t-2)} + 10\frac{\sin(1000\pi t)}{\pi t}$$

PROBLEM 13.3*:

(a) One approach is to first obtain the Fourier series of $x(t)$: ($f_o = 5$ Hz, $\omega_o = 10\pi$ rad/sec)

$$a_k = \frac{1}{0.2}\int_{-0.05}^{0.05}(1)e^{-j(10\pi)t}dt = \frac{1}{\pi k}\sin((\pi/2)k)$$

As a result, we get the following sum

$$x(t) = \sum_{k=-\infty}^{\infty}\frac{1}{\pi k}\sin((\pi/2)k)e^{j(10\pi)kt)}$$

$$x(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{\pi k}\sin((\pi/2)k)\cos((10\pi)kt)$$

The Fourier Transform consists of impulses at the harmonic frequencies:

$$X(j\omega) = \sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty}\frac{2}{k}\sin(k\pi/2)\delta(\omega - 10\pi k)$$

Since $\sin(k\pi/2)$ is zero for k even, the impulses at multiples of 20π will drop out; except for the impulse at $\omega = 0$ which has an area of π.

(b) We plot the input signal, the filter transfer function ($H(j\omega)$), and the output signal on the next page.

(c) Since frequencies, ω, of magnitude greater than 25π rad/sec are eliminated, we have the following output Fourier transform:

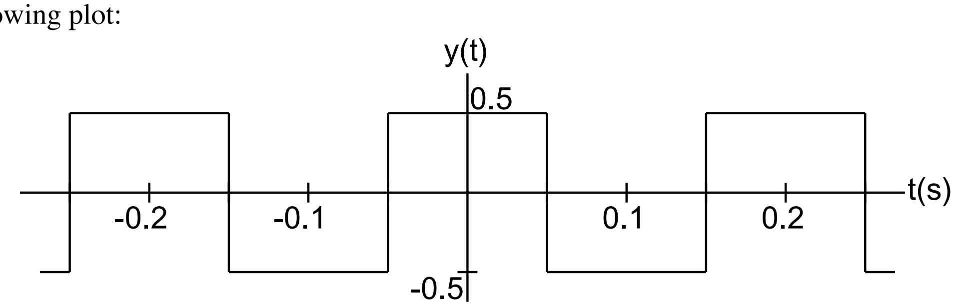
$$Y(j\omega) = \pi\delta(\omega) + 2\delta(\omega - 10\pi) + 2\delta(\omega + 10\pi)$$

Then we do the inverse transform of the 3 terms:

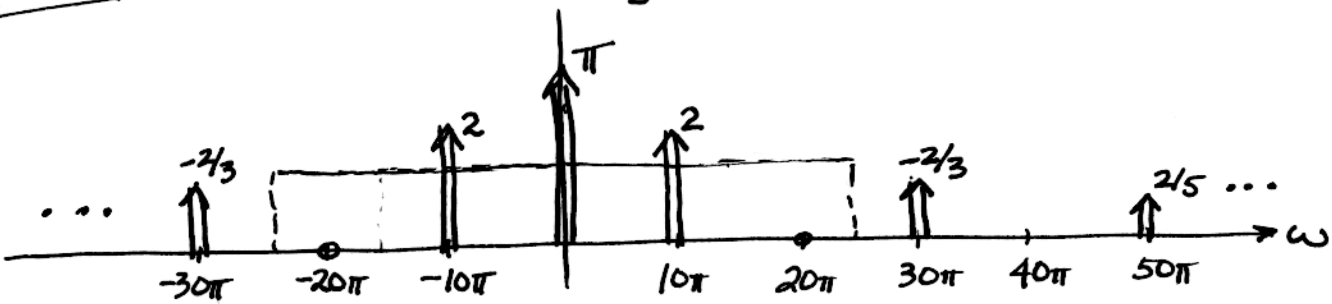
$$y(t) = \frac{\pi}{2\pi} + \frac{2}{2\pi}e^{j10\pi t} + \frac{2}{2\pi}e^{-j10\pi t} = \frac{1}{2} + \frac{2}{\pi}\cos(10\pi t)$$

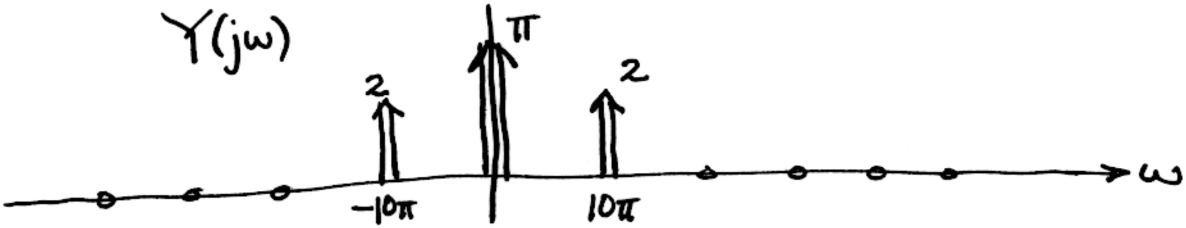
(d) We need to remove DC, so a highpass filter will do the trick. Its cutoff frequency must be less than 10π. We plot one such frequency response on the next page.

(e) If we remove the DC component from the input signal $x(t)$, we get $x(t) - \frac{1}{2}$ which is shown in the following plot:

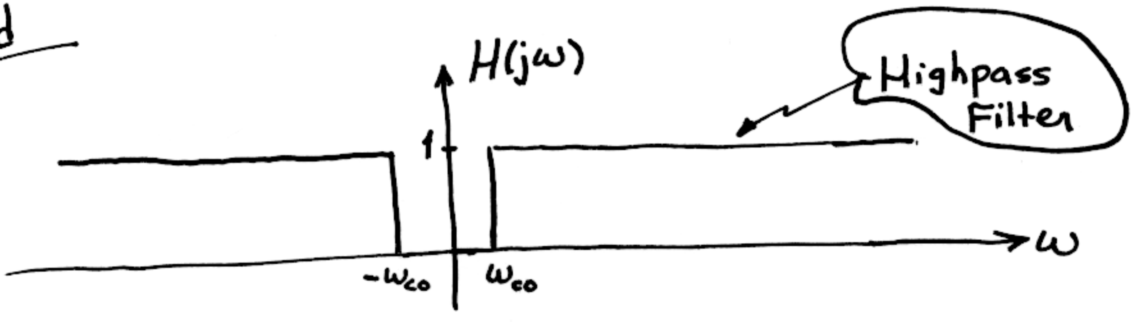


13.3 b





13.3d



NOTE: ω_{co} should be $0 < \omega_{co} < 10\pi$

PROBLEM 13.4*:

(a) Since we have the signal

$$w(t) = x_1(t)\cos(\omega_c t) + x_2(t)\sin(\omega_c t),$$

we can express the cosine and sine in exponential form according to Euler's formula:

$$w(t) = x_1(t)\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right) + x_2(t)\tfrac{1}{2}\left(-je^{j\omega_c t} + je^{-j\omega_c t}\right)$$

and then we get four terms in the Fourier transform:

$$W(j\omega) = \tfrac{1}{2}X_1(\omega-\omega_c) + \tfrac{1}{2}X_1(\omega+\omega_c) - j\tfrac{1}{2}X_2(\omega-\omega_c) + j\tfrac{1}{2}X_2(\omega+\omega_c)$$

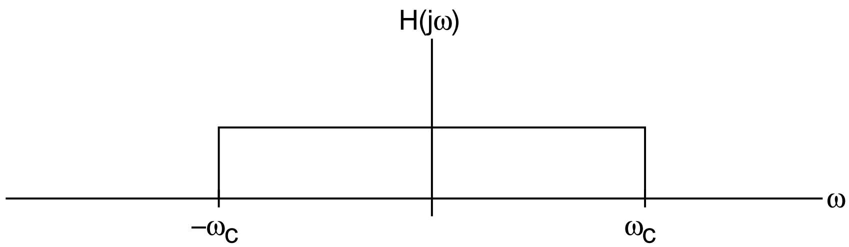
We sketch this function on the next page, assuming the Fourier transform of $x_1(t)$ is square, and $x_2(t)$ is triangular. The $\pm j$ that multiplies $X_2(j\omega)$ is illustrated by showing the triangle upside down in the positive frequency region to convey the fact that it is multiplied by $-j$, and also that its sign is different from the negative frequency component.

(b) In this case, the lower bounds for this frequency response, ω_a, is $\omega_c - \omega_m$, and the upper bounds for this frequency response, ω_b, is $\omega_c + \omega_m$.

(c) We can express $v(t)$ as

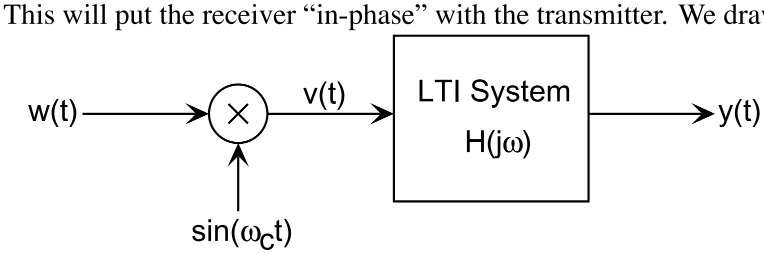
$$\begin{aligned}
v(t) &= \cos(\omega_c t)\left(x_1(t)\cos(\omega_c t) + x_2(t)\sin(\omega_c t)\right)\\
&= x_1(t)\cos(\omega_c t)\cos(\omega_c t) + x_2(t)\cos(\omega_c t)\sin(\omega_c t)\\
&= x_1(t)\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right) + x_2(t)\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)\tfrac{1}{2}\left(-je^{j\omega_c t} + je^{-j\omega_c t}\right)\\
&= x_1(t)\tfrac{1}{4}\left(e^{j2\omega_c t} + 2 + e^{-j2\omega_c t}\right) + x_2(t)\tfrac{1}{4}\left(-je^{j2\omega_c t} + (j-j) + je^{-j2\omega_c t}\right)\\
&= \tfrac{1}{2}x_1(t)\left(1+\cos(2\omega_c t)\right) + \tfrac{1}{2}x_2(t)\sin(2\omega_c t)
\end{aligned}$$

(d) We want a low-pass filter to eliminate the higher frequency components at $2\omega_c$; and also have a gain of 2 to restore the amplitude. One easy choice is to have a low-pass filter with a cutoff at the carrier frequency: $\omega_{\text{co}} = \omega_c$. We plot this response below.

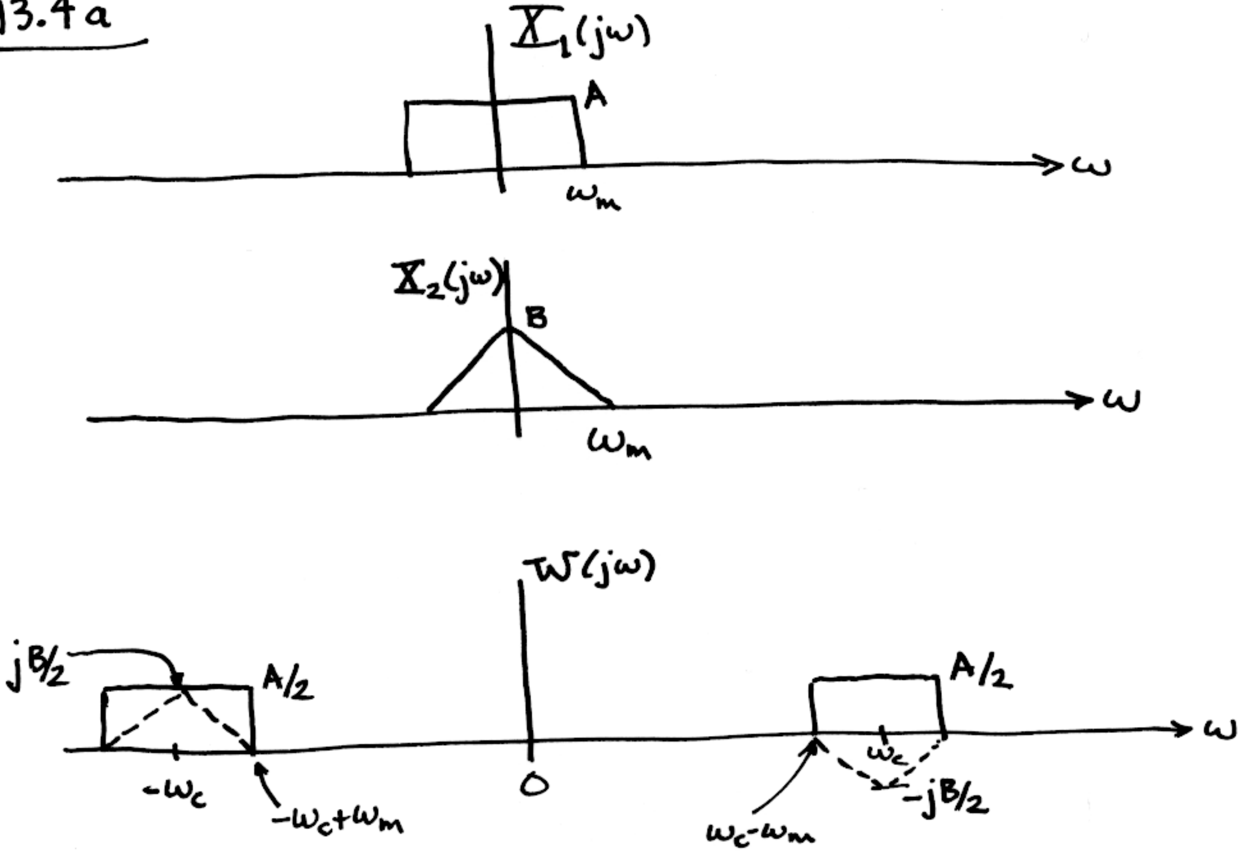


Other choices for the cutoff of the LPF would be anything greater than ω_m and less than $2\omega_c - \omega_m$.

(e) To recover the $x_2(t)$ signal, we will change the mixer to multiply by $\sin(\omega_c t) = \cos(\omega_c t - \pi/2)$, instead of $\cos(\omega_c t)$. This will put the receiver "in-phase" with the transmitter. We draw this system below.



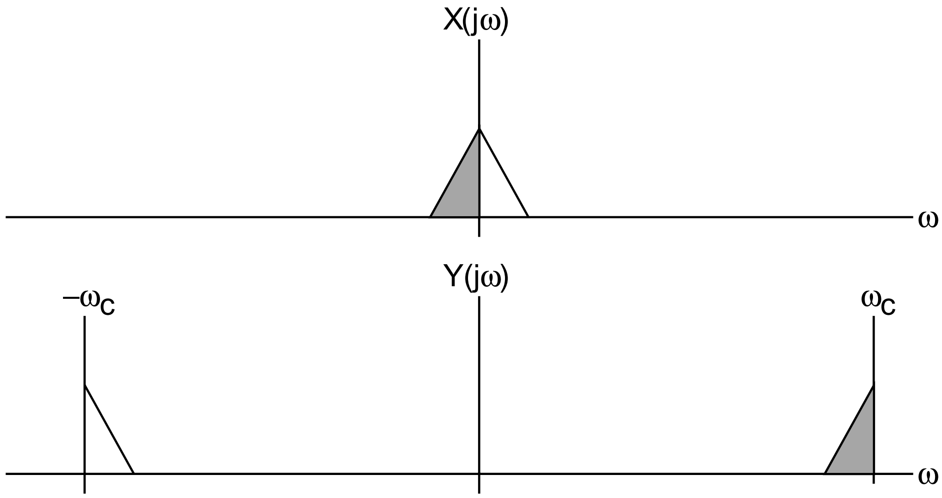
13.4 a



Note: $W(j\omega)$ has both a real part and an imaginary part. The triangle-shaped F.T. from $X_2(j\omega)$ becomes the imaginary part <u>if</u> $X_2(j\omega)$ is a purely real Fourier transform.

PROBLEM 13.5*:

(a) The cosine modulator moves the Fourier transform $X(j\omega)$ to be centered at $\pm 20000\pi$. Then the BPF slices out the spectral region from 19600π to 20000π and from -19600π to -20000π. We plot the resulting Fourier Transform of the output below, where $\omega_c = 20000\pi$.

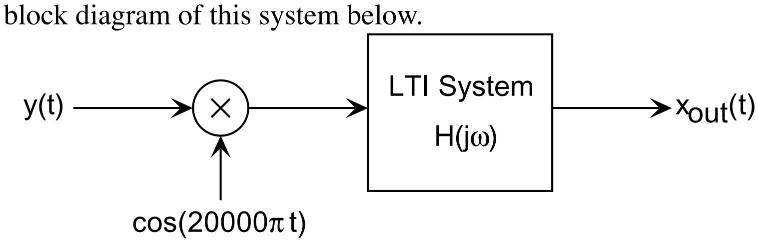


Notice that the negative frequency portion of $X(j\omega)$ appears in $Y(j\omega)$ in the spectral region from 19600π to 20000π; and the positive frequency portion of $X(j\omega)$ appears in $Y(j\omega)$ in the spectral region from -19600π to -20000π.

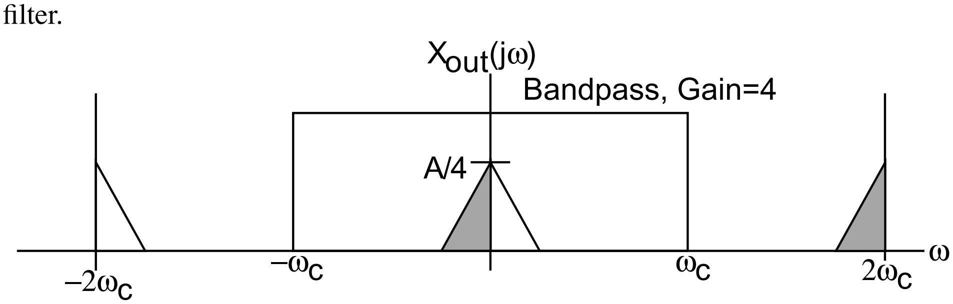
(b) To build a filter that gives the upper-sideband signal, we build an ideal bandpass filter with a gain of 2 between ω between 20000π and 20400π and -20000π and -20400π, and a gain of zero elsewhere.

(c) To get back the original signal, we modulate the incoming signal by $\cos(20000\pi t)$, and then bandpass this signal with an ideal lowpass filter of gain 4 and a cutoff frequency of $10kHz$, or 20000π rad/sec. The difference between demodulating a single sideband signal and a two sideband signal in this manner is that demodulating a single sideband signal gives half of the output signal power (therefore our lowpass filter has a gain of 4).

We show the block diagram of this system below.



We show the resulting signal when modulated by $\cos(20000\pi t)$ below along with the required bandpass filter.



(d) The resulting signal for g(t) can be calculated using the definition of the inverse Fourier transform:

$$
\begin{aligned}
g(t) &= \frac{1}{2\pi}\int_0^{20\pi}(7e^{j\omega t})d\omega \\
&= \frac{7}{2j\pi t}\left(e^{j20\pi t}-1\right) \\
&= \frac{7}{\pi t}\,e^{j10\pi t}\,\frac{e^{j10\pi t}-e^{-j10\pi t}}{2j} \\
&= \frac{7}{\pi t}\sin(10\pi t)e^{j10\pi t} \\
&= 7\,\frac{\sin(10\pi t)}{\pi t}\,e^{j10\pi t}
\end{aligned}
$$

Notice that this time signal is *complex-valued,* which happens because its Fourier transform $G(j\omega)$ is not conjugjate symmetric about $\omega = 0$.