GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #14

Assigned: 3 December 99
Due Date: Never

The Final Exam Time depends on the time of your scheduled lecture: Period 9, Dec. 15 (Wed.) for the 11am section, and Period 13, Dec. 17 (Fri.) for the 12pm section.
*****You must take the exam for your scheduled lecture section.**

Reading: Chapter 12 and all of Chapter 13.

$\Longrightarrow$ The five (5) **STARRED** problems do not have to be turned in for grading; however, problems of this type will appear on the **final exam**.

These will be covered in Recitation, and a solution will be posted after Friday, December 10.

PROBLEM 14.1:

Practice Problems: Look at Problems on Problem Sets 5 and 6 of EE3230, Winter, Spring, Fall of 1998, Problem Set 6 of EE2823A Winter 1999, and Problem Set 6 of EE2201 Spring 1999.

PROBLEM 14.2*:

The *scaling* property of the Fourier Transform gives an interesting relationship between time and frequency. If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then the scaling property states:

$$x(\beta t) \longleftrightarrow (1/|\beta|)X(j\omega/\beta)$$

Where β is a constant. The proof involves a simple change of variables, although the negative β case has to be looked at separately to get the absolute value in $(1/|\beta|)$. Here is the positive β case:

$$\int_{-\infty}^{\infty} x(\beta t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega\lambda/\beta}d\lambda/\beta = (1/\beta)\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/\beta)\lambda}d\lambda$$

(a) If $p(t)$ is a rectangular pulse of duration 1 sec: $p(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$

Draw a sketch of $q(t) = p(5t)$. Is $q(t)$ shorter or longer?

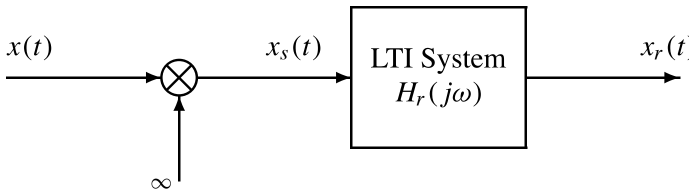
(b) The Fourier transform of a pulse is a "sinc" function; and the width of a "sinc" can be measured as the distance between the first zero crossings on either side of the mainlobe at $\omega = 0$. Determine the width of $P(j\omega)$.

(c) Use the scaling property to find the Fourier transform of $q(t)$, and then determine its width. Is $Q(j\omega)$ wider or narrower than $P(j\omega)$? Measure between the first zero crossings.

(d) The example above seems to obey the following statement: "the duration of a time signal is inversely proportional to the width of its Fourier transform." Commment on whether or not this is always true.

PROBLEM 14.3*:

Consider the following impulse train sampling and reconstruction system:



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

(a) If the input is $x(t) = 5 + 6\cos(20\pi t) + 10\cos(80\pi t - \pi/4)$, plot the Fourier transform $X(j\omega)$.

(b) Use the Sampling Theorem to determine the *maximum* value of T_s in the impulse train $p(t)$, so that there will be no aliasing when the input is $x(t)$ defined in part (a). In addition, determine the Nyquist rate for sampling this input signal.

(c) For the input in part (a), how should we choose the sampling rate $\omega_s = 2\pi/T_s$ so that $x_r(t) = x(t)$? Plot $X_s(j\omega)$ for the value of the sampling rate $\omega_s = 2\pi/T_s$ that is two times the Nyquist rate.

(d) If $\omega_s = 2\pi/T_s = 140\pi$ rad/sec. and $x(t)$ is the signal defined in part (a), the condition of the Sampling Theorem is not satisfied and aliasing distortion occurs. Plot the Fourier transform $X_s(j\omega)$ for this case.

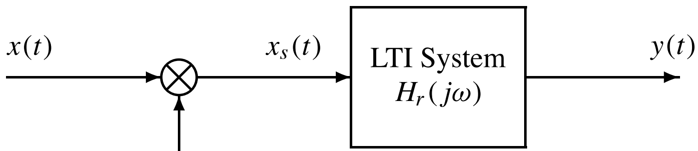
(e) Using the input signal from part (a) and the sampling rate of $\omega_s = 2\pi/T_s = 140\pi$ rad/sec., determine the Fourier transform $X_r(j\omega)$ of the output if the frequency response of the LTI system is

$$H_r(j\omega) = \begin{cases} T_s & |\omega| \le \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

(f) From part (e) determine the corresponding output time signal $x_r(t)$. Give a simple formula in terms of cosines.

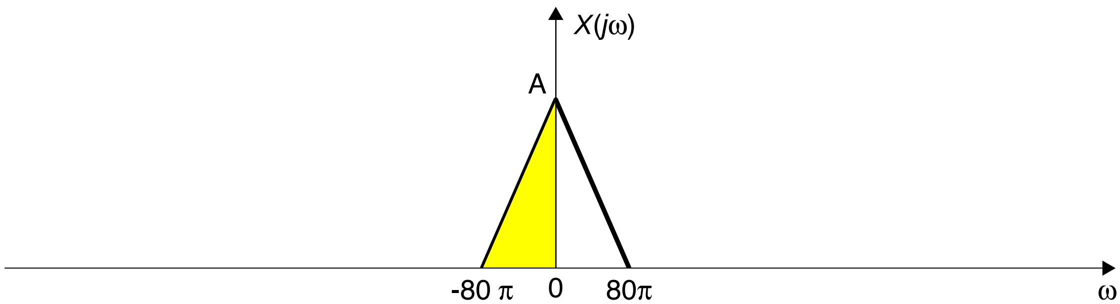
PROBLEM 14.4*:

The Sampling Theorem involves the operations of impulse train sampling and reconstruction as shown in the following system:



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The "typical" bandlimited Fourier transform of the input is depicted below:



(a) For the input with Fourier transform depicted above, use the Sampling Theorem to choose the sampling rate $\omega_s = 2\pi/T_s$ so that $x_r(t) = x(t)$? Plot $X_s(j\omega)$ for the value of $\omega_s = 2\pi/T_s$ that is two times the Nyquist rate.

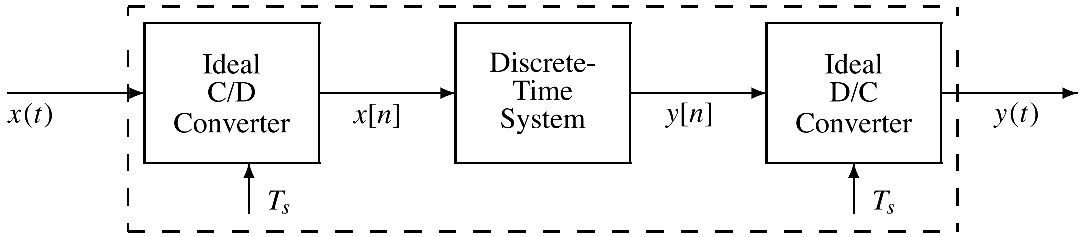
(b) If $\omega_s = 2\pi/T_s = 140\pi$ in the above system and $X(j\omega)$ is as depicted above, plot the Fourier transform $X_s(j\omega)$ and show that aliasing occurs. There will be an infinite number of shifted copies of $X(j\omega)$, so indicate what the pattern is versus ω.

(c) For the conditions of part (b), determine and sketch the Fourier transform of the output $X_r(j\omega)$ if the frequency response of the LTI system is

$$H_r(j\omega) = \begin{cases} T_s & |\omega| \leq \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

PROBLEM 14.5*:

All parts of this problem are concerned with the following system.



$$x[n] = x(nT_s) \qquad\qquad y(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

In all parts of this problem, assume that the input signal $x(t)$ is bandlimited so that $X(j\omega) = 0$ for $|\omega| \geq 80\pi$.

(a) Suppose that the discrete-time system is defined by $y[n] = x[n]$. What is the *minimum* value of the sampling frequency $\omega_s = 2\pi/T_s$ such that $y(t) = x(t)$?

(b) If the input is $x(t) = 5 + 6\cos(20\pi t) + 10\cos(80\pi t - \pi/4)$, the sampling frequency is $\omega_s = 140\pi$, and $y[n] = x[n]$, plot $Y(j\omega)$ and give an equation for $y(t)$. *Hint: Use the result of Problem 14.2(b).*

(c) For the rest of the parts, assume that the input/output relation for the discrete-time system is

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

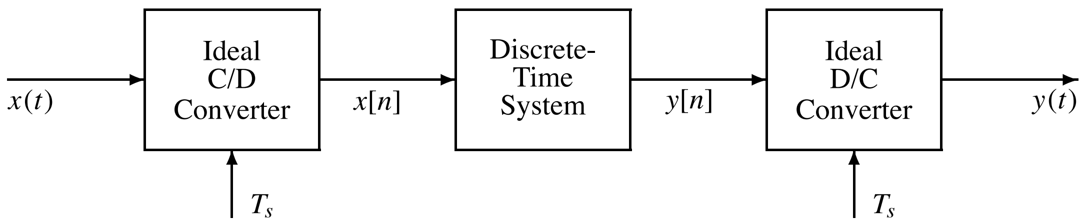
Make a plot of its frequency response $H(e^{j\hat{\omega}})$ versus $\hat{\omega}$.

(d) If $\omega_s = 2\pi/T_s$ is large enough so that there is no aliasing of the input, and the filter is defined in part (c), the input and output Fourier transforms are related by an equation of the form $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$. If $\omega_s = 200\pi$, find an equation for the overall effective frequency response $H_{\text{eff}}(j\omega)$, and plot $|H_{\text{eff}}(j\omega)|$ and $\arg[H_{\text{eff}}(j\omega)]$.
**Note: the "effective frequency response" $H_{\text{eff}}(j\omega)$ is only defined for $|\omega| < \frac{1}{2}\omega_s$ because the ideal D/C converter contains an ideal lowpass filter that is zero for $|\omega| > \frac{1}{2}\omega_s$.

(e) For the sampling rate $\omega_s = 200\pi$, and the filter of part (c) and the input signal of part (b), determine the output signal $y(t)$.

PROBLEM 14.6*:

All parts of this problem are concerned with the following system.



$x[n] = x(nT_s)$

$$y(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Assume that the input signal $x(t)$ is bandlimited,[1] so that $X(j\omega) = 0$ for $|\omega| \geq 1000\pi$.

(a) Suppose that the discrete-time system is defined by $y[n] = x[n]$. What is the *minimum* value of the sampling frequency $\omega_s = 2\pi/T_s$ such that $y(t) = x(t)$?

(b) Determine the relationship between $y[n]$ and $x[n]$ so that if the sampling rate satisfies the condition of (a), then $y(t) = x(t - 100T_s)$.

(c) The input/output relation for the discrete-time system is

$$y[n] = 0.8y[n-1] + 0.2x[n].$$

Make a plot of its frequency response $H(e^{j\hat{\omega}})$ versus $\hat{\omega}$.

(d) For the value of ω_s chosen in part (a) and the digital filter in part (c), the input and output Fourier transforms are related by an equation of the form $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$. Find an equation for the overall effective frequency response $H_{\text{eff}}(j\omega)$.
**Note: the "effective frequency response" $H_{\text{eff}}(j\omega)$ is only defined for $|\omega| < \frac{1}{2}\omega_s$ because the ideal D/C converter contains an ideal lowpass filter that is zero for $|\omega| > \frac{1}{2}\omega_s$.

(e) Suppose that no aliasing distortion occurs in sampling and that the frequency response of the discrete-time system is defined over one period ($-\pi \leq \hat{\omega} \leq \pi$) by

$$H(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| < \pi/4 \\ e^{-j\hat{\omega}5} & \pi/4 < |\hat{\omega}| \leq \pi \end{cases}$$

where $\hat{\omega} = \omega T_s$. Plot the magnitude and phase of $H_{\text{eff}}(j\omega)$, where $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$.

[1]The exact form of $X(j\omega)$ should not be needed to solve this problem, but if you find it necessary to draw a typical spectrum for $X(j\omega)$, use the triangular shape.