ECE-2025 Solutions P.S. #5. 17-Feb-00

Problem 5.1 Note that $a_{-k} = a_k^*$, hence $x(t)$ is a REAL signal.
Combine the terms for $k=-3$ and $k=3$, $k=-2$ and $k=2$, $k=-1$ and $k=1$. For $k=0$, the DC term is $\frac{1}{4}$.

i) Here we go:

$$a_1 e^{j2400\pi t} + a_{-1} e^{-j2400\pi t}$$

$$a_1 = \frac{1}{4+2j} = 0.2 - 0.1j = 0.2236\, e^{j\tan^{-1}(-0.5)} = 0.2236\, e^{-j\pi/6}$$

Remark! $\tan^{-1}(-0.5)$ does not really define the argument (phase angle) φ_1) uniquely.
Correct is:

$$\begin{cases} \sin\varphi_1 = \frac{-0.1}{0.2236} \\ \cos\varphi_1 = \frac{0.2}{0.2236} \end{cases} \rightarrow \varphi_1 = -\pi/6$$

Likewise:

$$a_2 = \frac{1}{4+4j} = 0.125 - 0.125j = 0.1768\, e^{j\tan^{-1}(-1)} = 0.1768\, e^{-j\pi/4}$$

(same caveat)

$$a_3 = \frac{1}{4+6j} = 0.0769 - 0.1154j = 0.1387\, e^{j\tan^{-1}(-1.5)} = 0.1387\, e^{-j0.9828}$$

(same caveat)

Noting that $\frac{X_k}{2} = a_k \Rightarrow$

$$\boxed{\begin{aligned} x(t) = 0.25 &+ 0.4472\cos(2400\pi t - 0.4636) + \\ &+ 0.3536\cos(4800\pi t - 0.7854) + \\ &+ 0.2774\cos(7200\pi t - 0.9828) \end{aligned}}$$

ii) The highest frequency is $\frac{7200\pi}{2\pi} = 3600\,\text{Hz}$.

By Nyquist's results, the minimum sampling rate is $3600 \times 2 = \underline{\underline{7200\,\text{Hz}}}$

Problem 5.2

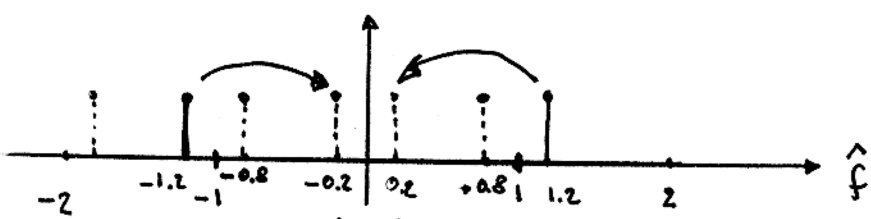
a) For 12 rpm, counter clockwise, the phasor (with reference frequency 0) is $e^{j\,12\times 2\pi t} = e^{j 24\pi t}$
(taking its magnitude (distance from axis of rotation) to be one)

b) For n flashes per second, the time between successive flashes is $\Delta t = \frac{1}{n}$ $(= T_s)$ Hence for apparent standstill an integer number of revolutions between successive flashes is needed, or $\frac{12}{n}$ must be an integer $\Rightarrow n = 1, 2, 3, 4, 6, 12.$
How does this problem relate to the course material on sampling?

$$\hat{\omega} = (24\pi).T_s = 2\pi \cdot \frac{12}{n} \quad \Rightarrow \quad \frac{12}{n} \text{ integer.}$$

c) $T_s = 100 \text{ msec} \quad \Rightarrow \quad x[k] = e^{j 24\pi.(k 0.1)} = e^{j 2.4 k\pi} = e^{j 0.4\pi k}$

d) $f_0 = 12$
$f_s = 10$ $\quad \rightarrow$ normalized frequency $= 1.2 = \hat{f}_0$
or $\hat{\omega} = 2.4\pi \text{rad}$



digital freq. (alias)

Problem 5.3

a) $$10\cos(0.25\pi n + 3\pi/4) = 10\cos\left(2\pi \cdot \frac{500n}{4000} + 3\pi/4\right)$$

$$= 10\cos\left(2\pi \cdot 500\, nT_{si} + 3\pi/4\right)$$

$$= 10\cos\left(2\pi \cdot 500\, t + 3\pi/4\right)\Big|_{t = nT_{si}}$$

$$\boxed{\rightarrow\ x_1(t) = 10\cos\left(2\pi \cdot 500t + 3\pi/4\right)}$$

Also

$$10\cos(0.25\pi n + 3\pi/4) = 10\cos(-0.25\pi n - 3\pi/4)$$

$$= 10\cos\left[(2\pi - 0.25\pi)\, n - 3\pi/4\right]$$

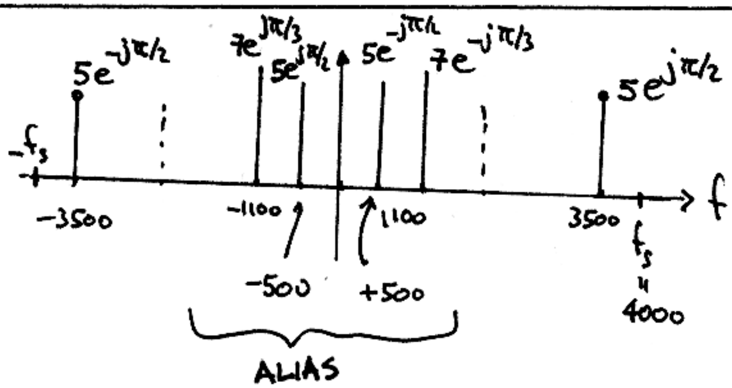
$$= 10\cos\left[2\pi \cdot \frac{3500}{4000}\, n - 3\pi/4\right]$$

$$= 10\cos\left[2\pi \cdot 3500\, nT_{si} - 3\pi/4\right]$$

$$= 10\cos\left[2\pi \cdot 3500\, t - 3\pi/4\right]\Big|_{t = nT_{si}}$$

$$\boxed{\rightarrow\ x_2(t) = 10\cos\left(2\pi \cdot 3500t - 3\pi/4\right)}$$

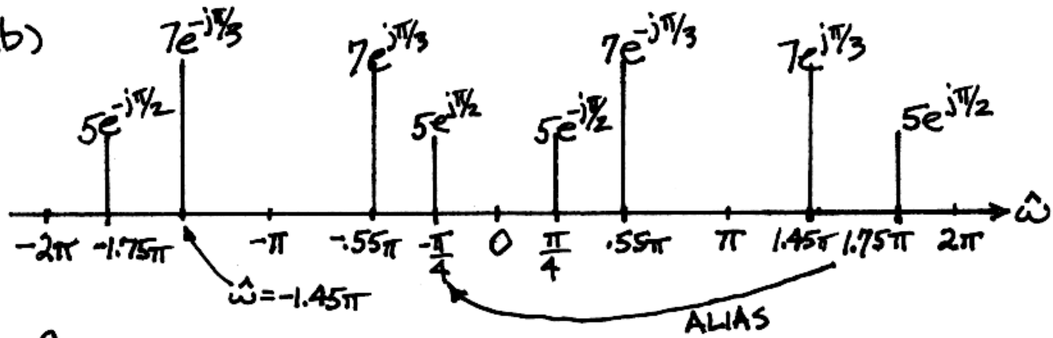
b)



Problem 5.3

discrete spectrum.

(b)



$f_s = 4000$ Hz

$f = 1100\text{Hz} \Rightarrow \quad \hat{\omega} = 2\pi\left(\frac{1100}{4000}\right) = 0.55\pi$ rads.

$f = 3500\text{Hz} \Rightarrow \quad \hat{\omega} = 2\pi\left(\frac{3500}{4000}\right) = 1.75\pi$ rads.

$\hat{\omega} = 1.75\pi$ rads. is EQUIVALENT to $\hat{\omega} = 1.75\pi - 2\pi = -0.25\pi$ rads

Also $\hat{\omega} = 0.55\pi$ rads is EQUIV. to $\hat{\omega} = 0.55\pi - 2\pi = -1.45\pi$ rads.

In the plot above, the range $-2\pi < \hat{\omega} < 2\pi$ is shown. This portrays the fact that the spectrum for the discrete-time signal is periodic with period $= 2\pi$

c) The converter D→C is twice as fast. This doubles all frequencies in the reconstruction.

∴ analog frequency components 1000 Hz
2200 Hz.

Prob 5.3

(c) The ideal D-C converter maps an $\hat{\omega}$ frequency back to f (Hz) via:

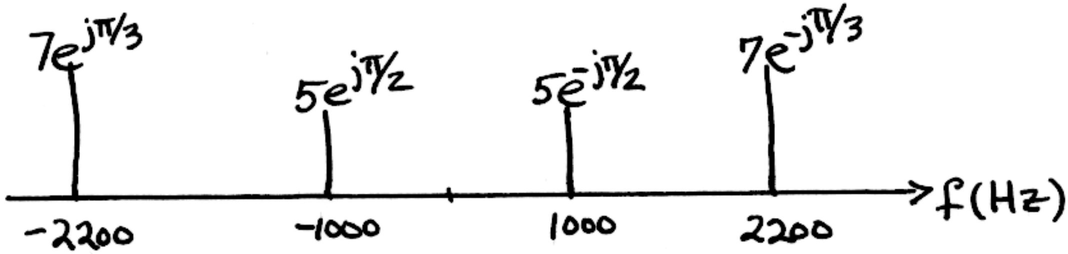
$$\hat{\omega} = 2\pi(f/f_s) \Longrightarrow f = \left(\frac{\hat{\omega}}{2\pi}\right) f_s$$

It also uses the lowest freqs, so it takes everything between $-\pi$ & π.

$$\hat{\omega} = \pm\pi/4 \longrightarrow f = \left(\frac{\pm\pi/4}{2\pi}\right) 8000 = \pm 1000 \text{ Hz}$$

$$\hat{\omega} = \pm 0.55\pi \longrightarrow f = \left(\frac{\pm 0.55\pi}{2\pi}\right) 8000 = \pm 2200 \text{ Hz}$$

The spectrum for the analog signal at the output of the C-D converter is:



The formula for the output signal is:

$$y(t) = 10\cos(2\pi(1000)t - \pi/2) + 14\cos(2\pi(2200)t - \pi/3)$$

Problem 5.4

a) $$x[n] = \mathrm{Re}\left(\sum_{i=1}^{4} (XX)_i \, e^{j\pi\hat{\omega}_i n} \right) \quad ; \quad n = 0 \text{ -- } 200000$$

$$= 1 + \mathrm{Re}\left[(1-j)\, e^{j\frac{\pi n}{4}}\right] + \mathrm{Re}\left[(-7j)\, e^{j\frac{3\pi n}{4}}\right] + \underbrace{\mathrm{Re}\left[(2j)\, \underset{\displaystyle e^{-j\frac{\pi}{4}n}}{\underset{\shortparallel}{e^{j\frac{7\pi}{4}n}}}\right]}_{= \mathrm{Re}\left[-2j\, e^{j\frac{\pi}{4}n}\right]}$$

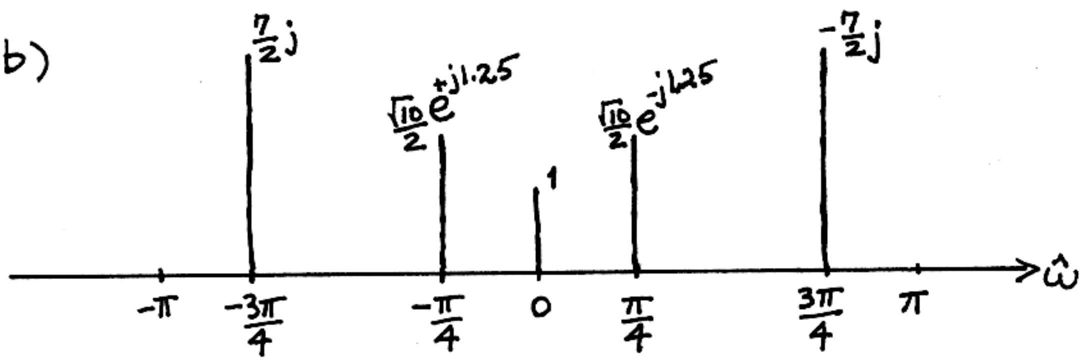
$$= 1 + \mathrm{Re}\left((1-3j)\, e^{j\frac{\pi}{4}n}\right) + \mathrm{Re}\left((-7j)\, e^{j\frac{3\pi}{4}n}\right)$$

$$= 1 + \sqrt{10}\cos\left(\frac{\pi}{4}n + \varphi\right) + 7\cos\left(\frac{3\pi}{4}n - \frac{\pi}{2}\right)$$

$\uparrow$

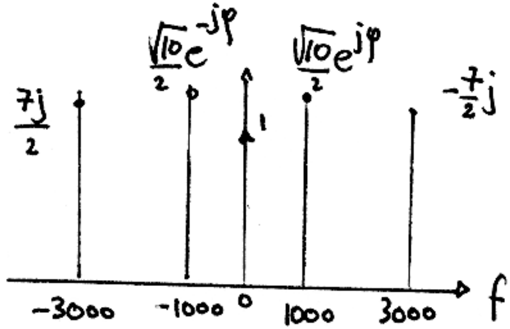
$\tan\varphi = -3 \quad ; \quad -\frac{\pi}{2} < \varphi < 0 \longrightarrow \varphi = -1.249$

(b)



Problem 5.5

a)



$\varphi = -1.249$

From $\hat{\omega} = 2\pi f . T_s \rightarrow f = \frac{\hat{\omega}}{2\pi T_s} = \frac{\hat{\omega}}{2\pi} f_s$

- DC.
- lower frequency: $f = \frac{\pi/4}{2\pi} 8000 = 1000$ Hz
- higher frequency: $f = \frac{3\pi/4}{2\pi} 8000 = 3000$ Hz

b) $x(t) = 1 + \sqrt{10}\cos(2\pi . 1000t + \varphi) + 7\cos\left(2\pi . 3000t - \frac{\pi}{2}\right)$

c) 200 000 samples total at 8000 samples/sec

duration $= \frac{200\,000}{8\,000} = \underline{25 \text{ sec}}$