GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Spring 2000
Problem Set #13

Assigned: 15-April-00
Due Date: 28-April-00

******* **CHANGE of PROCEDURE:** This homework is due no later than Noon on Friday, 28-April. You can turn it in at the last lecture.

Final Exam is scheduled for Tuesday, 2-May, at 11:30 AM. One page of hand-written material (two-sided) can be used on the exam. Coverage will be comprehensive, but the emphasis will be on the last half of the course.

Review Session for Final Exam: Monday, 1-May, at 6 PM.

Reading: In *DSP First*, Chapter 8 on *IIR Filters,* and review Chapter 7 on *Z-Transforms.*

$\Longrightarrow$ **Please check the "Bulletin Board" often. All official course announcements are posted there.**

The **STARRED** problems have to be turned in for grading. A solution will be posted to the web.

PROBLEM 13.1*:

The system function $H(z)$ and the impulse response $h[n]$ are two ways to define a LTI system. Use z-transform to answer the following:[1]

(a) Find the system function for $h_a[n] = u[n] - u[n-5]$.
Use the z-transform of $u[n]$ to express your answer as a ratio of polynomials in z^{-1}. Then simplify to get a polynomial in z^{-1} (i.e., no denominator). Is this an FIR or IIR filter?

(b) Find the system function for $h_b[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{2})^n u[n]$.
Express your answer as: (1) a sum of two first-order rational functions; and (2) a ratio of polynomials in z^{-1} (one numerator over one denominator).

(c) Determine the impulse response when the system function is: $H_c(z) = \dfrac{1 - z^{-2}}{1 + 0.5z^{-1}}$.

(d) Determine the impulse response when $H_d(z) = \dfrac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + 0.75z^{-1})}$.
Hint: write $H_d(z)$ as a sum of first-order rational functions.

(e) Determine the impulse response when $H_e(z) = 1 + 2z^{-2} + 4z^{-4} - 6z^{-6} - 8z^{-8}$.

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[1] A *rational* function is the ratio of two polynomials. For example, $H(z) = \dfrac{1 - z^{-1} + z^{-2}}{1 + 0.3z^{-1} + 0.4z^{-2}}$.

PROBLEM 13.2*:

We have developed several concepts that are useful in solving problems involving LTI systems. The main concepts are the *difference equation*, the *impulse response*, the *system function*, and the *frequency response* function. Most problem solving demands that you be able to go back and forth among these different mathematical representations of the LTI system because, as simple as it seems, the z-transform is *not* always the best tool for solving problems. Indeed, for a specific problem, one of these representations may be more convenient than the others, or we may need to use more than one of these representations in solving a given problem. The following is a simple problem that might be posed about an LTI system:

> *Given the input sequence $x[n]$ find the output sequence for all n when the system is an IIR filter:*
> $$y[n] = -0.5y[n-1] + x[n] - x[n-2].$$

The following is a partial list of possible approaches to solving this problem:

1. *Time-Domain:* Use the difference equation representation of the system to compute the output $y[n]$ for all required values of n.

2. *Z-Domain:* Multiply the z-transform of the input by the system function and determine $y[n]$ as the inverse z-transform of $Y(z)$.

3. *Frequency-Domain:* Break the input into a sum of complex exponential signals, use the frequency response function to determine the output due to each complex exponential signal separately, and finally, add the individual outputs together to get $y[n]$.

In each of these solution methods you would use one or more of the basic representations of the first-order IIR filter. Which method is easiest will have a lot to do with the nature of the input signal. For example, if you are given the difference equation and you want to use approach #2, you will have to determine the system function $H(z)$ from the difference equation coefficients.

Now in each of the following cases, the input will be given. In each case, determine which representation of the system and which of the above approaches will lead to the easiest solution of the problem, and detail the steps in using that approach to solve the problem. For example, if you choose approach #2 to solve the problem, your answer should be something like the following:

> **Step 1:** Find $X(z)$, the z-transform of $x[n]$.
>
> **Step 2:** Find $H(z)$, the system function of the first-order IIR filter.
>
> **Step 3:** Multiply $X(z)H(z)$ to get $Y(z)$.
>
> **Step 4:** Take the inverse z-transform of $Y(z)$ to get $y[n]$.

Now here are some possible inputs. In each case, state which of the approaches above (#1, #2, or #3) you would use. There may not be a clear cut answer. Give the approach that you *think* will yield the solution with least effort. Then carry out the method to get the output.

(a) $x[n] = u[n]$.

(b) $x[n] = \cos(0.1\pi n + \pi/2) + 10\cos(0.4\pi n - \pi)$ for $-\infty < n < \infty$.

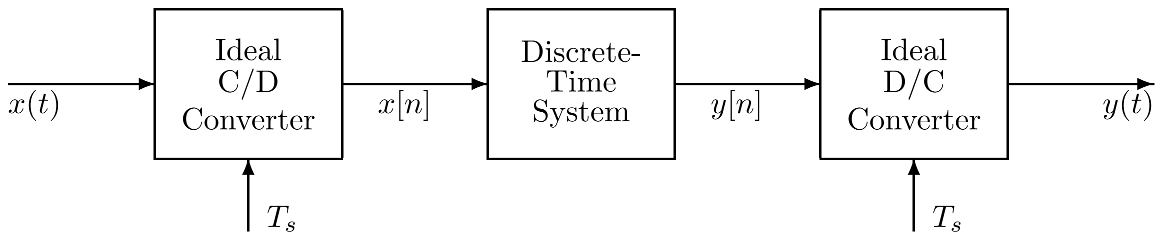
(c) $x[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{otherwise.} \end{cases}$

(d) $x[n] = -5\delta[n-20]$.

(e) $x[n]$ is a sampled speech signal. It is represented by a vector of 10000 numbers. In this case, you do not have to find the actual output.

PROBLEM 13.3*:

All parts of this problem are concerned with the following system.



$$x[n] = x(nT_s) \qquad\qquad y(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Assume that the input signal $x(t)$ is bandlimited,[2] so that $X(j\omega) = 0$ for $|\omega| \geq 1000\pi$.

(a) Suppose that the discrete-time system is defined by $y[n] = x[n]$. What is the *minimum* value of the sampling frequency $\omega_s = 2\pi/T_s$ such that $y(t) = x(t)$?

(b) The input/output relation for the discrete-time system is

$$y[n] = -\tfrac{1}{2} y[n-1] + 0.2x[n] - 0.2x[n-1].$$

Make a plot of its frequency response $H(e^{j\hat{\omega}})$ versus $\hat{\omega}$.

(c) For the value of ω_s chosen in part (a) and the digital filter in part (c), the input and output Fourier transforms are related by an equation of the form $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$. Find an equation for the overall effective frequency response $H_{\text{eff}}(j\omega)$.
**Note: the "effective frequency response" $H_{\text{eff}}(j\omega)$ is only defined for $|\omega| < \frac{1}{2}\omega_s$ because the ideal D/C converter contains an ideal lowpass filter that is zero for $|\omega| > \frac{1}{2}\omega_s$.

(d) Suppose that the sampling rate is high enough so that no aliasing distortion occurs in sampling and that the frequency response of the discrete-time system is defined over one period ($-\pi \leq \hat{\omega} \leq \pi$) by

$$H(e^{j\hat{\omega}}) = \begin{cases} e^{-j2\hat{\omega}} & |\hat{\omega}| < \pi/3 \\ 0 & \pi/3 < |\hat{\omega}| \leq \pi \end{cases}$$

where $\hat{\omega} = \omega T_s$. Plot the magnitude and phase of $H_{\text{eff}}(j\omega)$, where $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$. Note: this answer does *not* depend on the input signal, but if you want a specific value for f_s, then use $f_s = 1500$ Hz.

[2] The exact form of $X(j\omega)$ should not be needed to solve this problem, but if you find it necessary to draw a typical spectrum for $X(j\omega)$, use the triangular shape.

PROBLEM 13.4*:

An LTI system has the following system function:

$$H(z) = \frac{1 - z^{-2}}{1 + 0.5z^{-1}}.$$

The following questions cover most of the ways available for analyzing IIR discrete-time systems.

(a) Plot the poles and zeros of $H(z)$ in the z-plane.

(b) Determine the difference equation that is satisfied by the general input $x[n]$ and the corresponding output $y[n]$ of the system.

(c) Use z-transforms to determine the impulse response $h[n]$ of the system; i.e., the output of the system when the input is $x[n] = \delta[n]$.

(d) Determine an expression for the frequency response $H(e^{j\hat{\omega}})$ of the system.

(e) Use the frequency response function to determine the output $y_1[n]$ of the system when the input is

$$x_1[n] = 2\cos(\pi n) \qquad -\infty < n < \infty.$$

(f) Use the z-transform to determine the output $y_2[n]$ when the input is

$$x_2[n] = 2\cos(\pi n)u[n] = \begin{cases} 2(-1)^n & n \geq 0 \\ 0 & n < 0. \end{cases}$$

PROBLEM 13.5*:

For each of the difference equations below,[3] determine the poles and zeros of the corresponding system function, $H(z)$. Plot the poles **(X)** and zeros **(O)** in the complex z-plane.

$$\mathcal{S}_1: \quad y[n] = 0.4y[n-1] + x[n] + x[n-1]$$

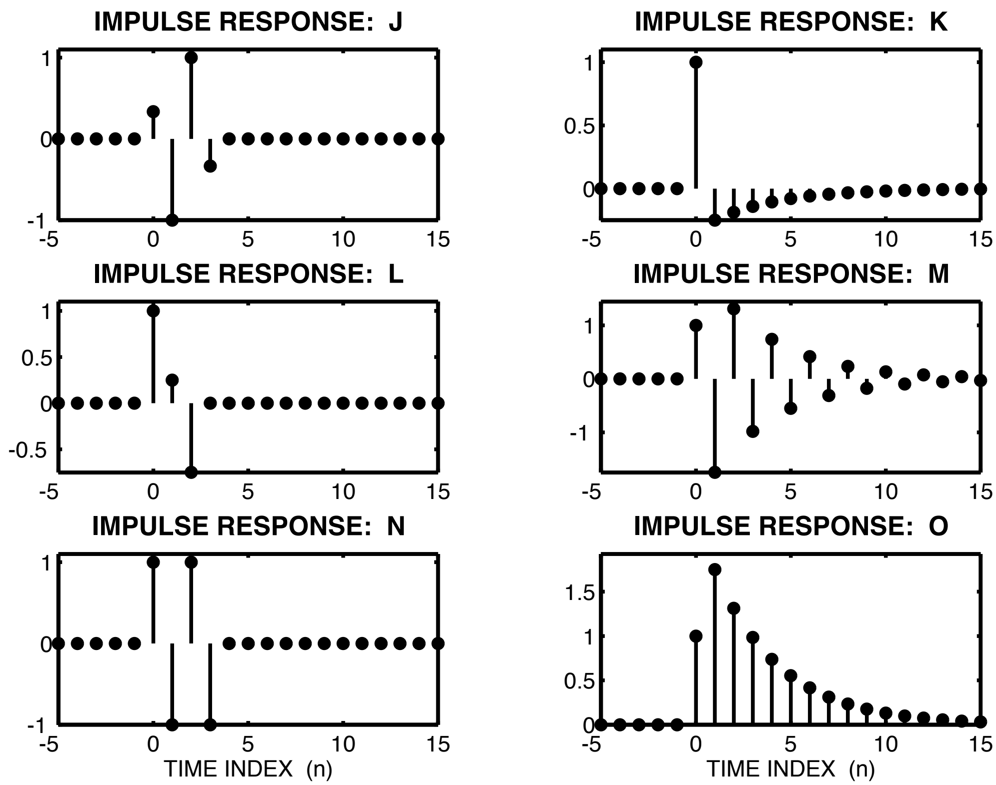
$$\mathcal{S}_3: \quad y[n] = -0.75y[n-1] + x[n] - x[n-1]$$

$$\mathcal{S}_6: \quad y[n] = x[n] - x[n-1] + x[n-2] - x[n-3]$$

$$\mathcal{S}_7: \quad y[n] = x[n] + \tfrac{1}{4}x[n-1] - \tfrac{3}{4}x[n-2]$$

[3] These systems are a subset of those in the following two problems. Furthermore, these matching problems cover a variety of FIR and IIR systems with different representations.

PROBLEM 13.6:



For each of the impulse-response plots (J, K, L, M, N, O), determine which one of the following systems[4] (specified by either an $H(z)$ or a difference equation) matches the impulse response. In addition, derive a formula for the impulse response, $h[n]$, for $\mathcal{S}_1$ and $\mathcal{S}_4$.

$\mathcal{S}_1$: $y[n] = 0.4y[n-1] + x[n] + x[n-1]$

$\mathcal{S}_2$: $H(z) = \dfrac{1+z^{-1}}{1-0.75z^{-1}}$

$\mathcal{S}_3$: $y[n] = -0.75y[n-1] + x[n] - x[n-1]$

$\mathcal{S}_4$: $H(z) = \dfrac{1-z^{-1}}{1-0.75z^{-1}}$

$\mathcal{S}_5$: $y[n] = x[n] - x[n-1] + x[n-2]$

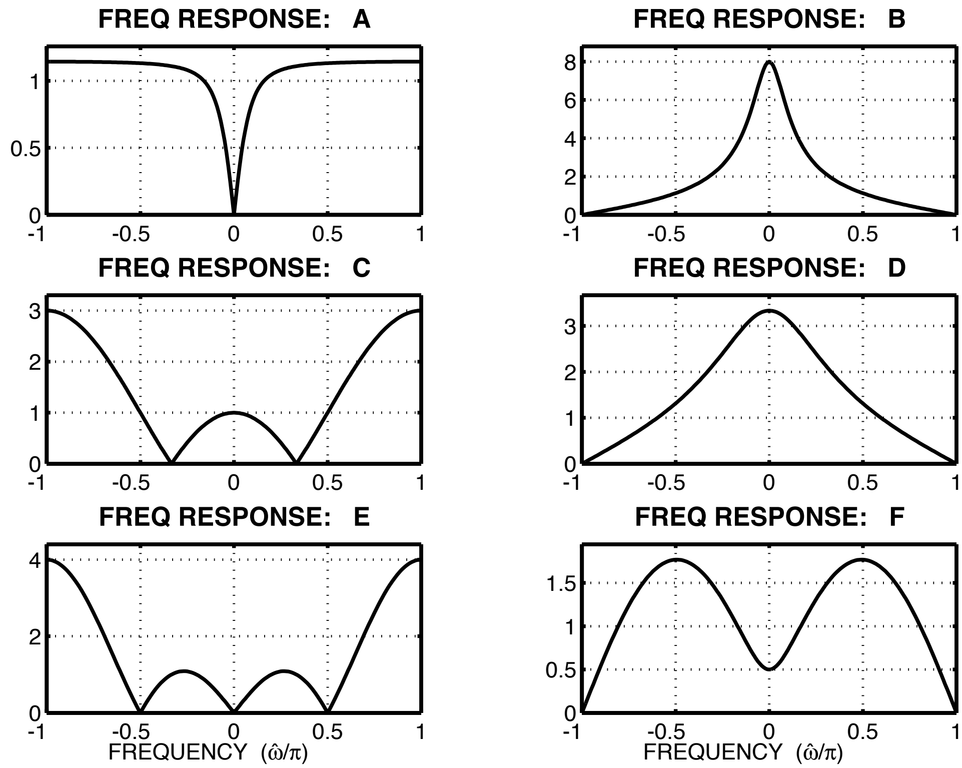
$\mathcal{S}_6$: $H(z) = 1 - z^{-1} + z^{-2} - z^{-3}$

$\mathcal{S}_7$: $y[n] = x[n] + \frac{1}{4}x[n-1] - \frac{3}{4}x[n-2]$

$\mathcal{S}_8$: $H(z) = \frac{1}{3}(1-z^{-1})^3$

[4]These 8 systems are exactly the same as the other matching problems.

PROBLEM 13.7:



For each of the frequency response plots (A, B, C, D, E, F), determine which one of the following systems[5] (specified by either an $H(z)$ or a difference equation) matches the frequency response (magnitude only). NOTE: frequency axis is **<u>normalized</u>**; it is $\hat{\omega}/\pi$. In addition, derive a formula for the magnitude-squared of the frequency response, $|H(e^{j\hat{\omega}})|^2$, for $\mathcal{S}_3$ and $\mathcal{S}_4$.

$\mathcal{S}_1:\quad y[n] = 0.4y[n-1] + x[n] + x[n-1]$

$\mathcal{S}_2:\quad H(z) = \dfrac{1+z^{-1}}{1-0.75z^{-1}}$

$\mathcal{S}_3:\quad y[n] = -0.75y[n-1] + x[n] - x[n-1]$

$\mathcal{S}_4:\quad H(z) = \dfrac{1-z^{-1}}{1-0.75z^{-1}}$

$\mathcal{S}_5:\quad y[n] = x[n] - x[n-1] + x[n-2]$

$\mathcal{S}_6:\quad H(z) = 1 - z^{-1} + z^{-2} - z^{-3}$

$\mathcal{S}_7:\quad y[n] = x[n] + \frac{1}{4}x[n-1] - \frac{3}{4}x[n-2]$

$\mathcal{S}_8:\quad H(z) = \frac{1}{3}(1-z^{-1})^3$

[5]These 8 systems are exactly the same as the other matching problems.