GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

**ECE 2025 Fall 1999**
**Lab #10: Design with Fourier Series**

Date: 3–6 April 2000

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This is *the official* Lab #10 description.

The lab report for this lab will be **INFORMAL:** discuss your Fourier Series results from section 4. The questions are very specific. Staple the **Instructor Verification** sheet to the end of your lab report.

The report will **due during the week of 10 April at the start of your lab.**

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# 1 Introduction & Objective

The goal of this laboratory project is to show that Fourier Series analysis is a powerful method for predicting the response of a LTI system when the input is a periodic signal. Since we will be doing Fourier Series for continuous-time signals, the formulas are integrals. As a result we will use MATLAB's numerical integration capability to calculate the Fourier Series coefficients of the output and the input; this method was introduced in Lab #9.

In this particular lab, we will use Fourier Series and the Fourier transform to analyze a power supply design problem in the frequency domain.

# 2 Background: Fourier Series Analysis and Synthesis

Recall the *analysis* integral and *synthesis* summation for the Fourier Series expansion of a periodic signal $x(t) = x(t + T_0)$. The Fourier synthesis equation for a periodic signal $x(t) = x(t + T_0)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{1}$$

where $\omega_0 = 2\pi/T_0$ is the *fundamental* frequency. To determine the Fourier series coefficients from a periodic signal, we must evaluate the *analysis* integral for every integer value of $k$:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt \tag{2}$$

where $T_0 = 2\pi/\omega_0$ is the *fundamental* period. If necessary, we can evaluate the analysis integral over any period; in (2) the choice $[-\frac{1}{2}T_0, \frac{1}{2}T_0]$ was a convenient one, but integrating over the interval $[0, T_0]$ would also give exactly the same answer.

The Fourier Series representation is extremely useful when studying the effects of an LTI filter, because the output signal is also periodic. The Fourier Series coefficients of the output signal $\{b_k\}$ are obtained by multiplying by the frequency response:

$$b_k = a_k \, H(j\omega_0 k) \tag{3}$$

where $H(j\omega_0 k)$ is the frequency response of the LTI system evaluated at the harmonics.

# 3 Warmup

In this project, we will use the Fourier Series coefficients to predict the response of a LTI system. Since you have already developed the capability to produce the Fourier Series numerically, the primary point of the warm-up is to show that you can adapt your existing MATLAB functions to do a new example quickly. In other words, you should utilize your results from Lab #9 to write functions that will be able to

1. Evaluate the Fourier Series coefficients for the following periodic square-wave signal which is defined over one period to be:

$$x(t) = \begin{cases} A & \text{for } 0 \le t \le 0.1T_0 \\ 0 & \text{for } 0.1T_0 < t < T_0 \end{cases} \tag{4}$$

   The amplitude $(A)$ and the length of the fundamental period $(T_0)$ should be parameters that can be varied.

2. Synthesize approximations to $x(t)$ using a finite number of Fourier Series coefficients $\{a_k\}$. This is similar to the sum of sinusoids function written in Lab #3.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi kt/T_0}$$

   where $2N+1$ is the number of terms used to form the signal and $T_0$ is the period.

3. Study and explain the convergence of the Fourier Series as $N \to \infty$.

If you are mathematically inclined, you might want to work out the Fourier Series integral for $x(t)$ in Eq. (4) by hand to show that the $\{a_k\}$ values are samples of a "sinc" function.

## 3.1 Square Wave Analysis

In this part of the warm-up, the objective is to make a plot of the spectrum for the square wave defined above.

(a) Since you will be using `quad8()` to do the integration, write the auxiliary function that defines the Fourier Series integrand for the square wave defined in Eq. (4).

$$x(t)e^{-j2\pi kt/T_0}$$

   Remember that it is necessary to introduce another parameter, the parameter $k$, when defining this auxiliary function.

(b) Write a MATLAB function that will evaluate the Fourier Series coefficients for the square wave over the range of indices $k = -N, \ldots, -1, 0, 1, 2, \ldots, N$. The function will contain a `for` loop to do all the coefficients from $k = -N$ to $k = +N$. It should return a vector containing $2N+1$ elements which are the $\{a_k\}$ coefficients.

(c) Use the auxiliary function written in the previous part to evaluate the Fourier Series coefficients for $x(t)$ from Eq. (4) for the case where $A = 25$, and $T_0 = 0.001$ secs. Find the $\{a_k\}$ coefficients for $N = 25$ and make a stem plot of the magnitude of the coefficients versus $k$.

   **Instructor Verification** (separate page)

### 3.2 Fourier Synthesis

In this part, you must synthesize approximations to $x(t)$ using a finite number of Fourier Series coefficients $\{a_k\}$. Let $N$ denote the largest index used, so that $2N+1$ is the number of terms used to form the signal.

(a) Using your `vsum_cos` function for sinusoidal synthesis from a finite number of terms, generate $x_N(t)$ for $N = 7$ and $N = 15$.[1] For each of these synthesized signals make a plot showing the synthesized signal and $x(t)$ on the same plot. Use a two-panel subplot to show the two cases on one page.

Reminder: You have to set up a time grid for the time interval, and you should use three periods of the signal from $t = -T_0$ to $t = 2T_0$. The spacing between grid points must be small enough so that the sampling rate (grid points per second) is at least *twice as high as the highest frequency component in your signal,* and even more oversampling will probably be needed to "see" convergence.

(b) Explain how you are getting convergence as $N$ increases. Where does the approximation error seem to be largest?

**Instructor Verification** (separate page)

### 3.3 Filtering a Periodic Signal

In the lab project, you will have to "filter" the periodic input signal through a continuous-time LTI system whose frequency response, so it will be necessary to plot frequency responses such as the following:

$$H(j\omega) = \frac{j\omega}{1000 + j\omega}$$

(a) Make a plot of the magnitude and phase f $H(j\omega)$ versus frequency. Use a frequency range that extends from $-5000$ Hz to +5000 Hz.

**Instructor Verification** (separate page)

(b) Determine the Fourier Series coefficients of the output signal $y(t)$ when the input is the periodic square wave defined in (4). Use the frequency response $H(j\omega)$ and apply (3). Make a plot of the (magnitude) spectrum of $y(t)$ over the frequency range $-5000$ Hz to +5000 Hz.

(c) Synthesize an approximation $y_N(t)$ by using $2N+1 = 11$ Fourier coefficients.

## 4 Analysis of a Power Supply Circuit

A common electrical design problem is that of converting an AC voltage to a DC voltage; no doubt you have in your possession many of these little "power packs" for modems, calculators, phones, etc. An alternating current (AC) voltage waveform is a sinusoid at 60 Hz (in the US); a direct current (DC) voltage is a constant, or zero frequency waveform. Figure 1 depicts a circuit for creating a DC voltage from an AC voltage. The diode bridge circuit implements what is called a "full-wave rectifier," and the RC circuit provides lowpass filtering to produce a (nearly) constant DC output.

In this lab exercise, you will analyze a power supply circuit (Fig. 1) and design its parameters by using methods in the frequency domain. The strategy consists of finding the Fourier Series for the signal $x(t)$;

[1] In order to use the `vsum_cos()` function for Fourier synthesis, we must include all the $\{a_k\}$ coefficients for both the positive and negative $k$. Likewise the frequency vector, `fk` must contain both positive and negative harmonics.

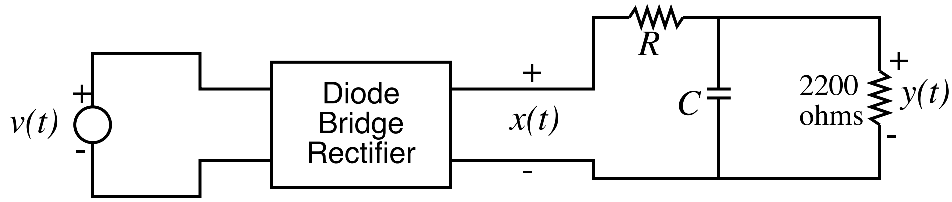


Figure 1: Power Supply circuit: AC to DC voltage converter

multiplying by the frequency response of the RC circuit (a lowpass filter); and then evaluating the size of the Fourier coefficients for the output signal. The diode-bridge rectifier is a full-wave rectifier defined by the equation

$$x(t) = |v(t)| \tag{5}$$

where $v(t)$ is the input and $x(t)$ is the output. The output of the rectifier, $x(t)$, is the input to the RC circuit and $y(t)$ is the output of that circuit. Once you have learned circuits, it would be easy to write the differential equation that describes the R-C circuit in Fig. 1. The result would be the following differential equation which gives the relationship between the input and output voltages of the circuit:

$$\frac{d}{dt}y(t) + \left(\frac{2200 + R}{2200RC}\right) y(t) = \frac{1}{RC}x(t) \tag{6}$$

In a real power supply, the signal $v(t)$ would be the 60-Hz powerline AC voltage, which would be represented mathematically as $v(t) = 120\sqrt{2}\cos(120\pi t)$. The units of $v(t)$ are *volts*.

If you take the Fourier transform of (6) and solve for the frequency response $H(j\omega)$ the result is

$$H(j\omega) = \frac{\beta}{j\omega + \alpha}$$

where the parameters $\alpha$ and $\beta$ can be written in terms of $R$ and $C$.

The block diagram shown in Fig. 2 represents the operations performed by the different parts of the circuit. The input-output equation for the full-wave rectifier is defined by $x(t) = |v(t)|$. The purpose of the rectifier is to generate a periodic signal with a non-zero DC component. The purpose of the lowpass filter is to remove most of the high frequencies in the output of the rectifier, leaving the DC component.

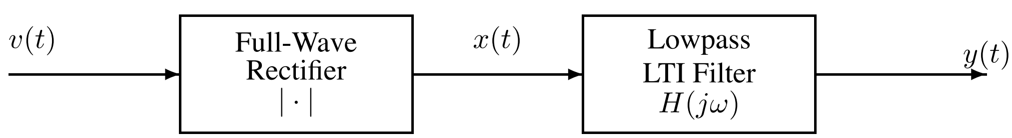


Figure 2: Block diagram representation of power supply.

## 4.1 Fourier Analysis of the Full-Wave Rectifier Output

Assume that the input to the rectifier is a power line voltage $v(t) = 120\sqrt{2}\cos(120\pi t)$.

(a) It is not too hard to verify that the true impulse response of the R-C circuit is a one-sided exponential:

$$h(t) = \frac{1}{RC}e^{-\alpha t}u(t)$$

where $\alpha = (2200 + R)/2200RC$. You just substitute $h(t)$ into the differential equation (6) and show that both sides match. It involves taking the first derivative and adding together the two terms on the left-hand side of the differential equation (6). Remember that the derivative of the unit-step signal, $u(t)$, is the unit impulse signal, $\delta(t)$. You should include this derivation in a lab report.

(b) To find the frequency response of the circuit, we can "take the Fourier transform" of $h(t)$ and get the frequency response of the lowpass filter in the following form:

$$H(j\omega) = \frac{\beta}{j\omega + \alpha}$$

Notice that the values of $\alpha$ and $\beta$ can be written in terms of $R$ and $C$; you will need this later on when evaluating the frequency response $H(j\omega)$.[2]

Write a MATLAB function that will plot the magnitude and phase of $H(j\omega)$ over the range $-2\pi(400) \leq \omega \leq 2\pi(400)$. For the input arguments, use $R$ and $C$. Write this code from scratch using `plot()` or `fplot()`; don't use MATLAB's built-in frequency response function.[3] Test your function by making the the plot for some typical values of $R$ and $C$ (e.g., $R = 33,000$ ohms and $C = 5 \times 10^{-6}$ farads).

(c) Define MATLAB expressions for the functions $v(t)$ and $x(t)$. The output of the full-wave rectifier is a periodic signal such that $x(t) = x(t + T_0)$. Determine the *fundamental* period $T_0$ of the rectified signal $x(t)$. Use `subplot( )` and `fplot( )` to make a plot showing both $v(t)$ and $x(t)$ over the range $0 \leq t \leq 3T_0$.

(d) Since the output of the full-wave rectifier, $x(t)$, is a periodic signal, it has a Fourier Series. Write the MATLAB code to evaluate the Fourier coefficients of $x(t)$ numerically with `quad8()`. Make the limits on the Fourier analysis integral $-\frac{1}{2}T_0$ to $\frac{1}{2}T_0$, because this gives the simplest form for the integration.

(e) You should also derive this expression as a mathematical formula to verify that you have the correct Fourier Series coefficients. You can derive this mathematical formula by hand—just break the cosine into two complex exponentials and grind out the integrals.

(f) Since $x(t)$ is periodic, it can be represented approximately by a truncated Fourier series. Use your formula for the Fourier coefficients $\{a_k\}$ for $k = 0, \pm1, \pm2, \ldots, \pm N$ to synthesize an approximation to $x(t)$; use $N = 10$ and call the resulting signal $x_{10}(t)$. This can be done by evaluating the numeric values of the Fourier coefficients and using your function `vsum_cos()` from an earlier lab.

Make a plot over the range $0 \leq t \leq 3T_0$ that compares the exact $x(t)$ to the approximate one, $x_{10}(t)$. At which times is $x_{10}(t)$ most different from the exact $x(t)$?

### 4.2 Find the Output Signal

The Fourier coefficients of the output signal are $b_k = a_k H(jk\omega_0)$, because the theory of the frequency response tells us how to determine the exact output of the lowpass filter by tracking each sinusoidal component through the filter: Using our $2N+1$ term approximation for the input, the approximate output is

$$y_N(t) = \sum_{k=-N}^{N} b_k e^{jk\omega_0 t} = \sum_{k=-N}^{N} a_k H(jk\omega_0) e^{jk\omega_0 t} \tag{7}$$

where the $a_k$ are the Fourier coefficients of $x(t)$.

[2]Alternate approach: We can "take the Fourier transform" of the differential equation; the derivative term becomes a multiplication by $j\omega$ in the frequency domain. and $H(j\omega)$ is found by dividing $Y(j\omega)/X(j\omega)$.

[3]You cannot use `freqz()` because this is not a digital filter; MATLAB has a function called `freqs()`, but you shouldn't use this function in this lab.

(a) *Frequency Domain:* Make a three-panel plot showing the spectrum of $x_N(t)$ in the top for $N = 3$; the magnitude of $H(j\omega)$ in the middle (use a frequency range that lines up with the top plot); and the **spectrum** of $y_N(t)$ in the bottom plot (for $N = 3$ also). To do this, you must have specific values for $R$ and $C$ to determine $\alpha$ and $\beta$ in the formula for $H(j\omega)$, so let $R = 33,000$ ohms and $C = 5 \times 10^{-6}$ farads in this part and the next.

(b) *Time Domain:* Next, you should make a plot of the output signal in the time domain for $N = 10$, i.e., plot $y_{10}(t)$ versus $t$ over the range $0 \leq t \leq 11T_0$.

This requires that you evaluate the $b_k$ Fourier coefficients numerically and use `vsum_cos` to create $y_N(t)$. In this approach, use the Fourier coefficients $a_k$ that were evaluated numerically, and then evaluate the frequency response $H(j\omega)$ at the appropriate frequencies. Then the product would be the Fourier coefficients of the output, as given by Eq. (3).

### 4.3 Design the Power Supply in the Frequency Domain

The power supply circuit could be solved in the time-domain to get the output waveform. If you had the signal $y(t)$, then you could measure the DC component of the output voltage, and you would also notice that the output signal contains an oscillating component called the *ripple*. The objective of the design is to control the size of the ripple by choosing $R$ and $C$ carefully.

(a) Refer to your Fourier Series formula above, and determine the DC value of the input $x(t)$. You should have already made a plot of $x(t)$, so mark the DC value of $x(t)$ on that plot.

(b) In this part, let the values of $R$ and $C$ be unknown parameters. Use your knowledge of the input Fourier series and $H(j\omega)$ to write a mathematical formula for the DC component of the output (in terms of $R$ and $C$). Hint: Use the frequency response $H(j\omega)$ to find the DC value of the first-order RC filter in terms of $R$ and $C$. As a sanity check, when you finally get the correct formula for the output DC term, it should not depend on $C$.

(c) When you made the plot of $y_{10}(t)$, you should have observed a "ripple" in the time-domain signal. The ripple in the output is due to all the non-DC terms in the input Fourier Series, but it is mostly due to the terms for $k = \pm 1$.

$$y(t) = b_0 + \sum_{k=1}^{\infty} \left( b_k e^{jk\omega_0 t} + b_{-k} e^{-jk\omega_0 t} \right)$$

Therefore, the output can be well approximated by considering the signal $y_1(t)$ that contains only one sinusoidal term plus DC. You should derive the mathematical formula for $y_1(t)$ in terms of the unknown parameters $R$ and $C$. However, you only need the mathematical formula for the *magnitude* of the first sinusoidal term in $y_1(t)$ (the phase is less important). In the process of doing this derivation you will have to determine the period of the ripple, so give the value of the period in secs.

(d) Now we can complete a general design of the power supply for any specification on the output DC voltage ($V_{\text{out}}$) and the ripple voltage ($V_r$). Suppose that our design specification is to have the DC component of the output be $V_{\text{out}} = 3.3$ volts with a ripple of $\pm V_r = \pm 0.15$ volts, i.e., peak-to-peak ripple of 0.3 volts. The design amounts to finding values for $R$ and $C$. In previous parts, we have written equations for the DC component and the first harmonic, so we can solve for $R$ and $C$

Now we will solve two non-linear equations in two unknowns to get formulas for $R$ and $C$ so that the DC component of the output is exactly $V_{\text{out}}$ volts, and the oscillating component of $y_1(t)$ satisfies the ripple specification, $\pm V_r$. Use your results from parts (b) and (c) to set up the two equations: one for

DC and the other for $k = 1$. Show that you can solve the DC equation for $R$ in terms of the desired $V_{\text{out}}$. Then plug that result into the second equation and rearrange to get a formula for $C$ in terms of the ripple $\pm V_r$, as well as $R$ and $V_{\text{out}}$. This second equation involves the magnitude of a complex quantity, $H(j\omega)$, so the algebra can be messy.

### 4.4 Power Supply Output Waveform

(a) Verify your formulas in the previous section by calculating the *exact* values of $R$ and $C$ needed to get $V_{\text{out}} = 3.3$ volts with a ripple of $\pm V_r = \pm 0.15$ volts.

(b) Make plots of the output signal to confirm that your design in the previous part is correct.
To verify your $R$ and $C$ values from the previous part, calculate the values of the output Fourier Series coefficients $\{b_k\}$, and make a plot of the output voltage signal when using $N = 1$ and $N = 10$ coefficients, i.e., plot $y_1(t)$ and $y_{10}(t)$. The reason for plotting $y_{10}(t)$ is that it should represent the true output, while $y_1(t)$ is an approximation because it uses only one Fourier coefficient.

(c) Do both plots exhibit a voltage ripple that stays between 3.15 and 3.45 volts? Explain why or why not. Explain the validity of using one Fourier coefficient for the design. How much error is introduced by ignoring $\{b_k\}$ for $k > 1$.

# Lab #10
# ECE-2025 Spring-2000
# Instructor Verification

Name: ________________________________________ Date of Lab: __________

Part 3.1 Illustrate Fourier analysis for a square wave. Find the $\{a_k\}$ coefficients numerically. Plot the spectrum of the square wave versus frequency.

Verified:____________________ Date/Time:__________

Part 3.2 Illustrate Fourier synthesis from the $\{a_k\}$ for a square wave. Make some plots with different numbers of coefficients. Explain convergence.

Verified:____________________ Date/Time:__________

Part 3.3 Plot the magnitude and phase of a continuous-time filter.

Verified:____________________ Date/Time:__________