EE-2025 Spring-2000

Lecture 6

Fourier Series Coefficients

31-Jan-00

Web-CT Info

- <u>Check the Bulletin Board for msgs</u>
- Old Quizzes & Problems are linked
  - Quiz #1 on 4-Feb (Friday)
- Prob Set #3 due This Week
  - Solution will be posted Thurs @ 6PM

Lab Info

- Lab #2 Report
  - <u>ERRATA ? ALWAYS Check Bulletin Board</u>
  - Turn in during your lab time
  - Write-up sections 4 and 5
  - Include INSTRUCTOR VERIFICATION
- Lab #3 was posted Friday
  - Learn some Music Notation

READING ASSIGNMENTS

- This Lecture:
  - <u>Notes on Fourier Series</u>
    - 17 pages, posted to WebCT
    - Replace pp 62-66 in Chapter 3
- Other Reading:
  - Next Lecture: Chap. 4 on Sampling

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- ANALYSIS via Fourier Series
  - For PERIODIC signals: x(t+T) = x(t)
  - SPECTRUM from the Fourier Series

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

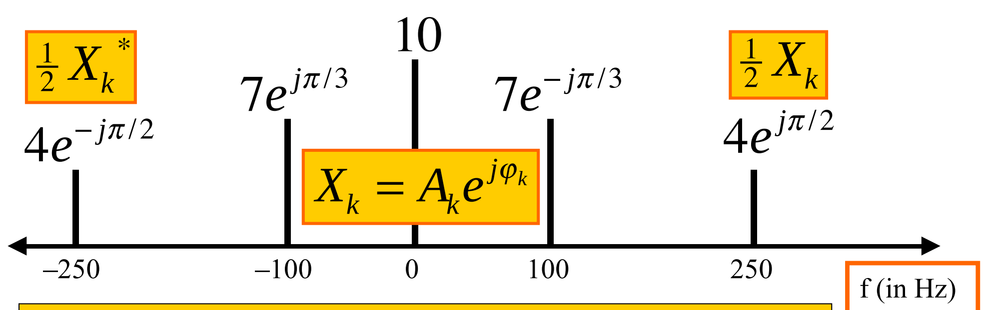
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = X_0 + \sum_{k=1}^{N}\left\{\tfrac{1}{2}X_k e^{j2\pi f_k t} + \tfrac{1}{2}X_k^* e^{-j2\pi f_k t}\right\}$$

Harmonic Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\omega_0 = \frac{2\pi k}{T_0} = \left(\frac{2\pi}{T_0}\right)k = 2\pi(f_0)k$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

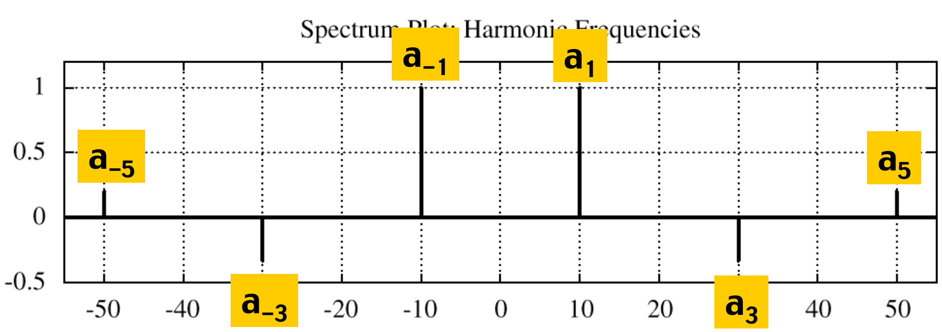
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



a_k is the complex amplitude for kf_0

SYNTHESIS vs. ANALYSIS

- **SYNTHESIS**
  - **Easy**
  - **Given (ω_k, A_k, ϕ_k) create x(t)**
- **Synthesis can be HARD**
  - **Synthesize Speech so that it sounds good**
- **ANALYSIS**
  - **Hard**
  - **Given x(t), extract (ω_k, A_k, ϕ_k)**
  - **How many?**
  - **Need algorithm for computer**

STRATEGY

- ANALYSIS
  - Get representation from the signal
  - Works for PERIODIC Signals
- Fourier Series
  - INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt$$

FUNDAMENTAL FREQ: $f_0 = 1/T_0$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi mt/T_0}\,dt = \frac{T_0}{-j2\pi m} e^{-j2\pi mt/T_0}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t}\,dt = 0 \qquad \omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

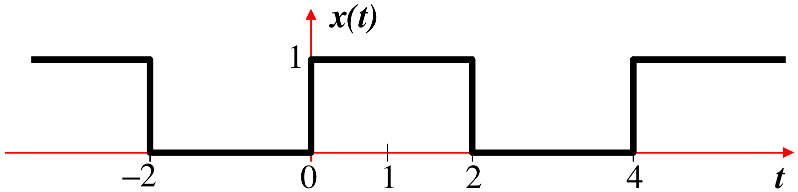
$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi \ell t/T_0} e^{-j2\pi kt/T_0}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(\ell - k)t/T_0}\,dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 4$:



FS for a SQUARE WAVE

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k \ne 0)$$

$$a_k = \frac{1}{4}\int_0^2 1e^{-j2\pi kt/4}dt = \tfrac{1}{4(-j\pi k/2)}e^{-j\pi kt/2}\Big|_0^2$$

$$= \tfrac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{AREA})$$

$$a_0 = \tfrac{1}{4}\int_0^2 dt = \tfrac{1}{4}(2-0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

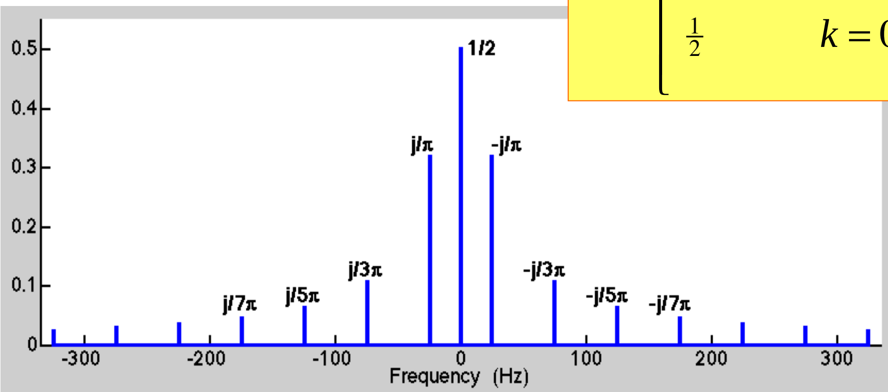
- a_k is a function of k
  - Complex Amplitude for k–th Harmonic
  - Does NOT depend on the period, T_0

$$a_k = \frac{1-e^{-j\pi k}}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm1, \pm3, \dots \\ 0 & k = \pm2, \pm4, \dots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

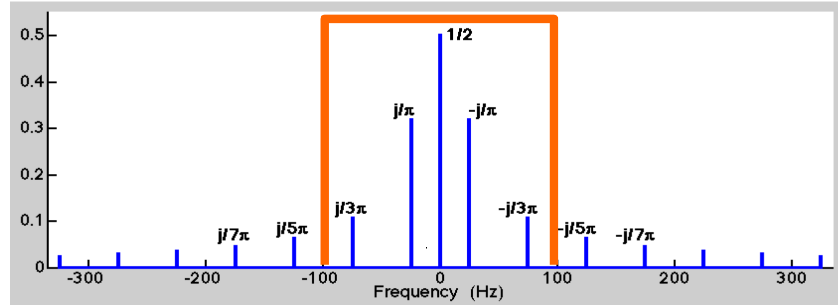
$\omega_0 = 2\pi(25)$

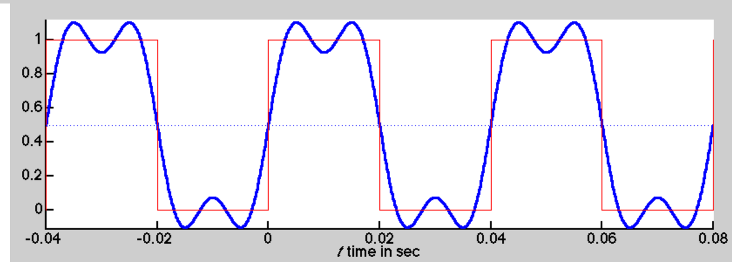
$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



SYNTHESIS: 3rd Harmonic

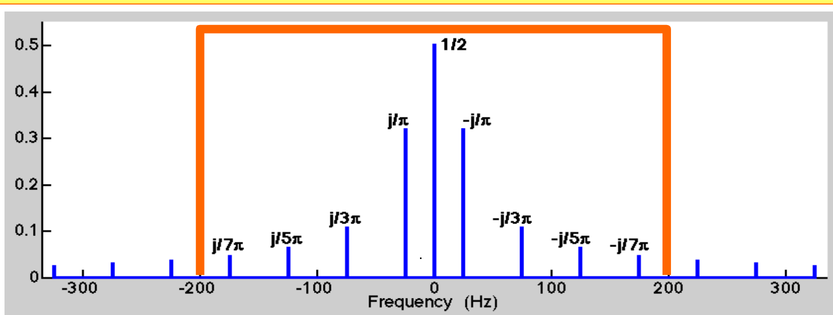
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

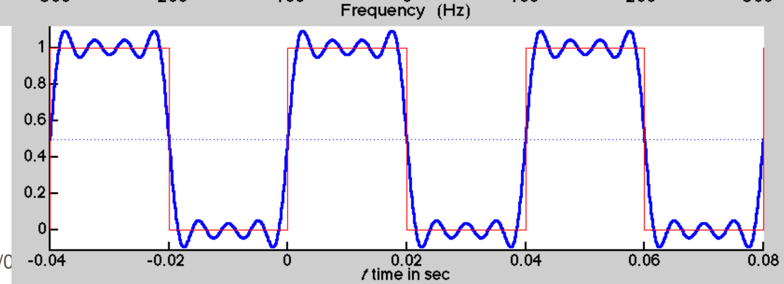




SYNTHESIS: 7th Harmonic

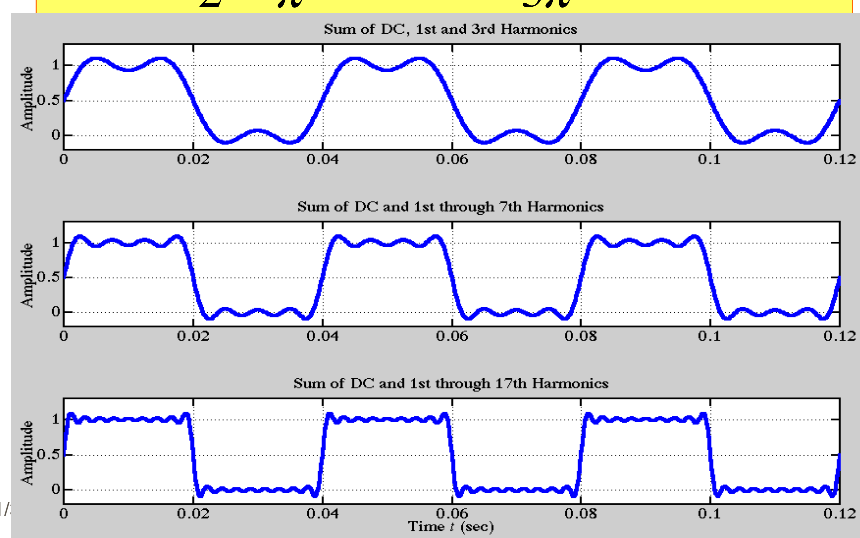
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





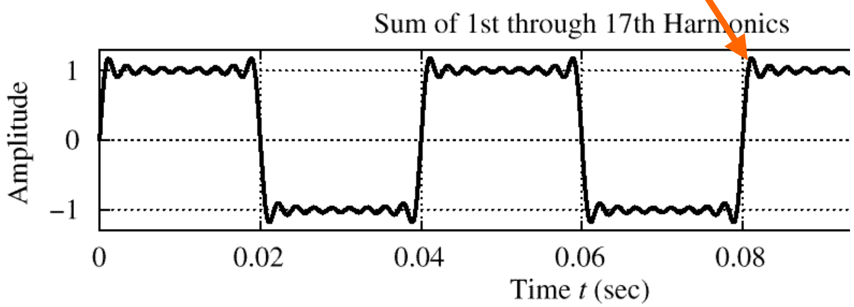
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at **DISCONTINUITY** of x(t)
  - There is always an **overshoot**
  - **9%** for the Square Wave case



A Couple of DEMOS

- Beat Control GUI
  - DSPFirst Toolbox: MATLAB
    - DSPFIRST/beatcon.m
- Fourier Series Java Applet
  - Interactive
  - http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html

Fourier Series Java Applet