EE-2025 Spring-2000

Lecture 14

Frequency Domain from H(z)

13-March-00

READING ASSIGNMENTS

- This Lecture:
  - Chapter 7, pp. 220–230
- Other Reading:
  - Recitation & Lab: Ch. 7, pp. 220–239
    - ZEROS (and POLES)
  - Next Lecture: Notes on Continuous-Time

LECTURE OBJECTIVES

- ZEROS and POLES
- Relate H(z) to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- **THREE DOMAINS:**
  - Show Relationship for FIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter (Find b_k)
  - Reject completely 0.7π, 0.8π, and 0.9π
  - Estimate the filter length needed to accomplish this task. How many b_k ?
- Z POLYNOMIALS provide the TOOLS

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $H(z) = \sum_{n} h[n] z^{-n}$

APPLIES to Any SIGNAL

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^2 + 2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

CONVOLUTION PROPERTY

- Convolution in the n–domain
  - SAME AS
- Multiplication in the z–domain

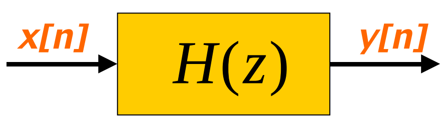
$$y[n] = h[n] * x[n] \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

MULTIPLY Z–TRANSFORMS

$$y[n] = h[n] * x[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR Filter

CONVOLUTION EXAMPLE



$$x[n] = \delta[n-1] + 2\delta[n-2] \qquad h[n] = \delta[n] - \delta[n-1]$$

$$y[n] = x[n] * h[n]$$

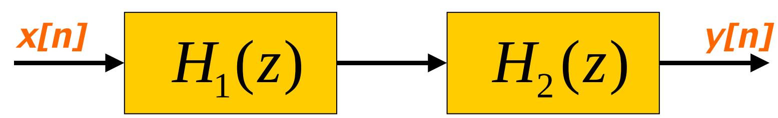
$$X(z) = z^{-1} + 2z^{-2} \qquad H(z) = 1 - z^{-1}$$

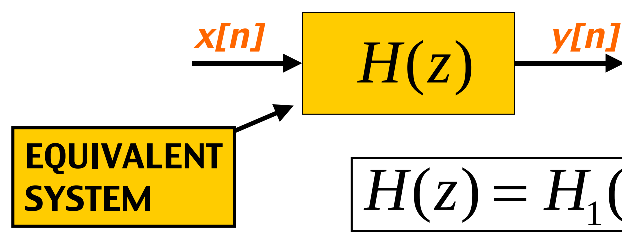
$$Y(z) = (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$$

$$y[n] = \delta[n-1] + \delta[n-2] - 2\delta[n-3]$$

CASCADE EQUIVALENT

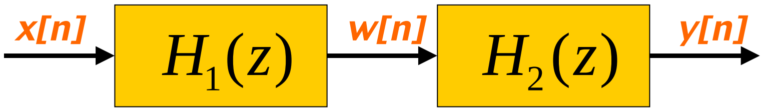
- Multiply the System Functions





$$H(z) = H_1(z)H_2(z)$$

CASCADE EXAMPLE



$$w[n] = x[n] - x[n-1]$$

$$y[n] = w[n] + w[n-1]$$

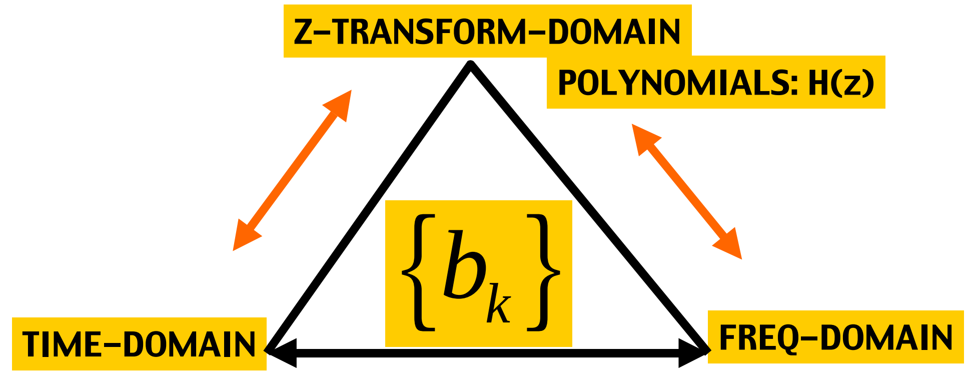
$$H_1(z) = 1 - z^{-1}$$

$$H_2(z) = 1 + z^{-1}$$

$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n-2]$$

THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

FREQUENCY RESPONSE ?

- **Same Form:**

$\hat{\omega}$ – Domain

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k \left(e^{j\hat{\omega}}\right)^{-k}$$

$$z = e^{j\hat{\omega}}$$

z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

CHANGE in NOTATION

- **Relate H(z) to FREQUENCY RESPONSE**

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- **NEW NOTATION for FREQUENCY RESPONSE**

$$H(\hat{\omega}) \leftrightarrow H(e^{j\hat{\omega}})$$

ANOTHER ANALYSIS TOOL

- z–Transform POLYNOMIALS are EASY !
  - ROOTS, FACTORS, etc.
- ZEROS and POLES: where is H(z) = 0 ?
- The z–domain is COMPLEX
  - H(z) is a COMPLEX–VALUED function of a COMPLEX VARIABLE z.

ZEROS of H(z)

- Find z, where H(z)=0

$$H(z) = 1 - \tfrac{1}{2} z^{-1}$$

$$1 - \tfrac{1}{2} z^{-1} = 0\ ?$$

$$z - \tfrac{1}{2} = 0$$

$$\text{Zero at:}\quad z = \tfrac{1}{2}$$

ZEROS of H(z)

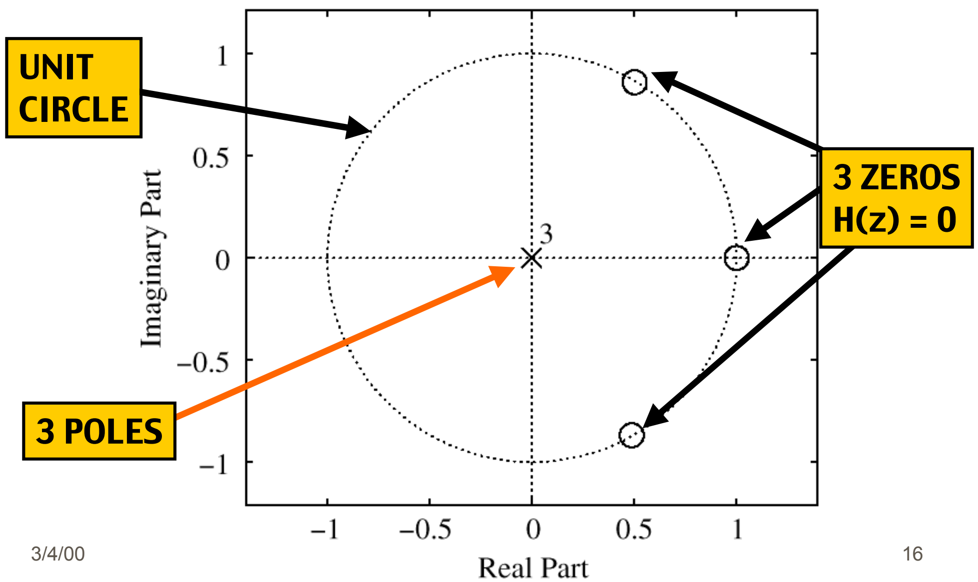
- Find z, where H(z)=0
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots:}\quad z = 1,\ \tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2} \qquad e^{\pm j\pi/3}$$

PLOT ZEROS in z-DOMAIN



POLES of H(z)

- Find z, where $H(z) \to \infty$
  - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

Three Poles at: $z = 0$

FREQ. RESPONSE from ZEROS

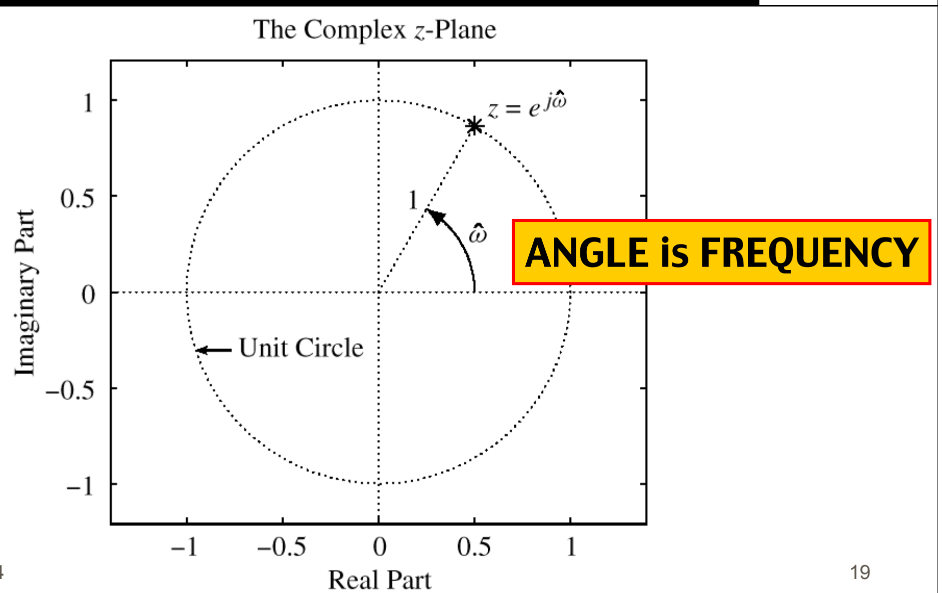
$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- Relate H(z) to FREQUENCY RESPONSE
- EVALUATE H(z) on the UNIT CIRCLE
  - ANGLE is same as FREQUENCY

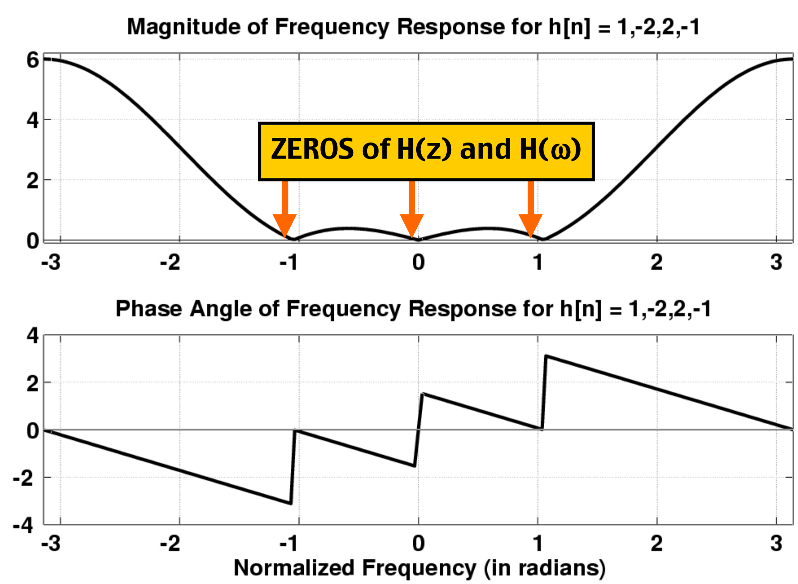
$z = e^{j\hat{\omega}}$ (as $\hat{\omega}$ varies)

defines a CIRCLE, radius $= 1$

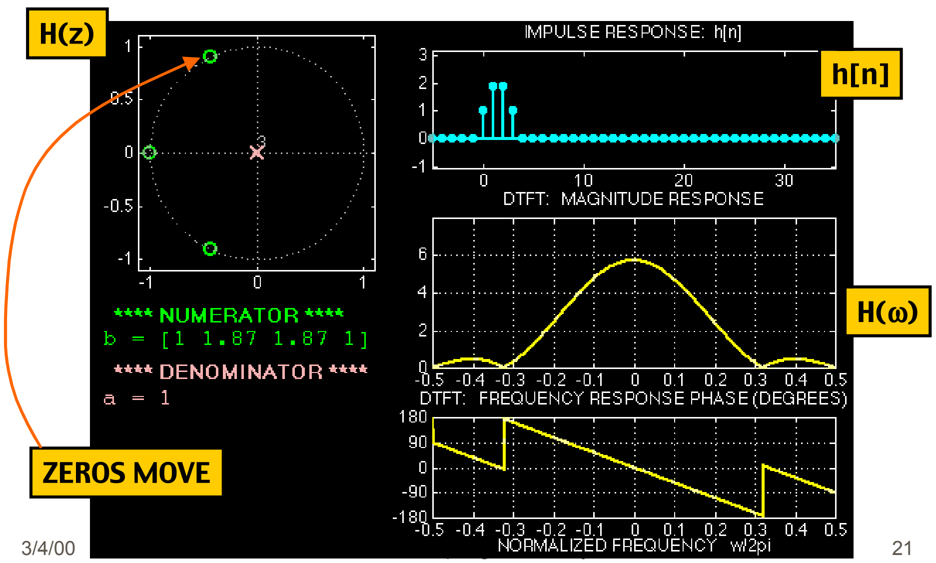
$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$



FIR Frequency Response



3 DOMAINS MOVIE: FIR

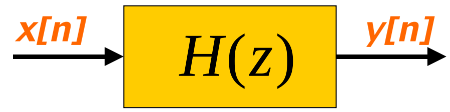


NULLING PROPERTY of H(z)

- When H(z)=0 on the unit circle.
  - Find inputs x[n] that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

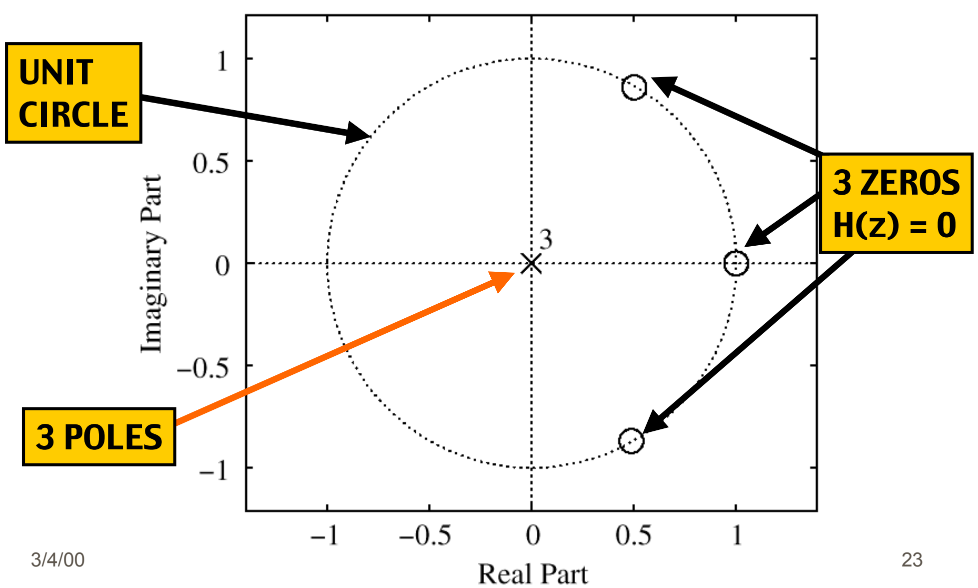


$$H(e^{j\pi/3}) = ?$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

PLOT ZEROS in z-DOMAIN



NULLING PROPERTY of H(z)

- Evaluate H(z) at the input "frequency"

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

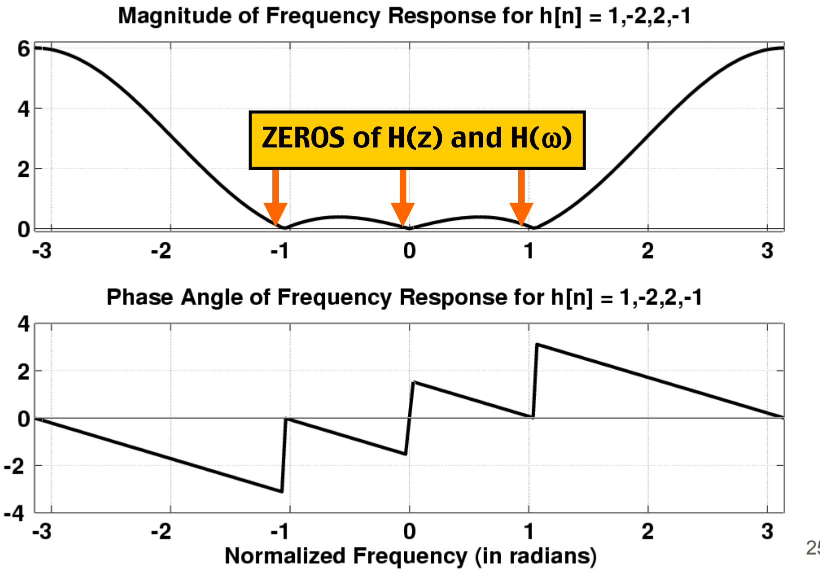
$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 1)$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response



L-pt RUNNING SUM H(z)

$$H(z)=\sum_{k=0}^{L-1}\tfrac{1}{L}z^{-k}=\frac{1-z^{-L}}{L(1-z^{-1})}=\frac{z^{L}-1}{Lz^{L-1}(z-1)}$$

$$z^{L}-1=0 \quad\Rightarrow\quad z^{L}=1=e^{j2\pi k}$$

$$z=e^{j(2\pi/L)k} \quad \text{for } k=1,2,\ldots L-1$$

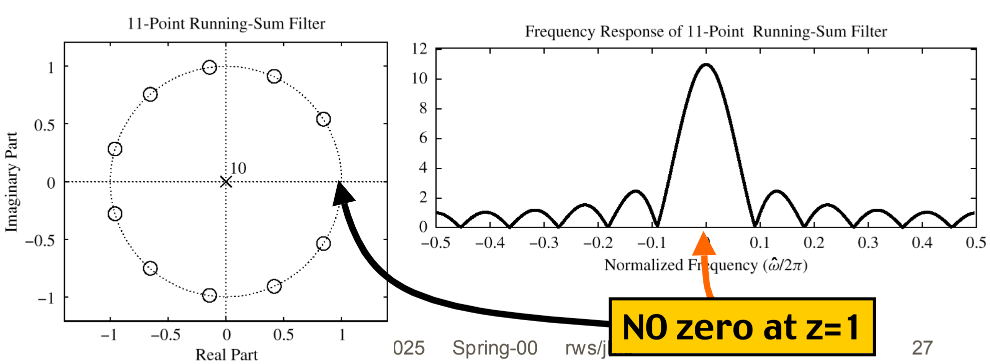
ZEROS on UNIT CIRCLE

(z–1) in denominator cancels k=0 term

11-pt RUNNING SUM H(z)

$$H(z)=\sum_{k=0}^{10}z^{-k}$$

$$H(z)=(1-e^{j2\pi/11}z^{-1})(1-e^{j4\pi/11}z^{-1})\cdots(1-e^{j20\pi/11}z^{-1})$$



FILTER DESIGN: CHANGE L