EE-2025 Spring-2000

Lecture 16

Convolution (Continuous-Time)

20-March-00

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 7–April
- Get NEW CHAPTERS
  - PDF or Bookstore
- Prob Set #8 is due this week
- Lab QUIZ THIS WEEK

READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, pp. 1020–1041
- Other Reading:
  - Recitation: Ch. 10, pp. 1020–1029
  - Next Lecture: Start reading Chapter 11

LECTURE OBJECTIVES

- Review of C–T LTI systems
- Evaluating convolutions
  - Examples
  - Impulses
- LTI Systems
  - Cascade and parallel connections
  - Stability and causality

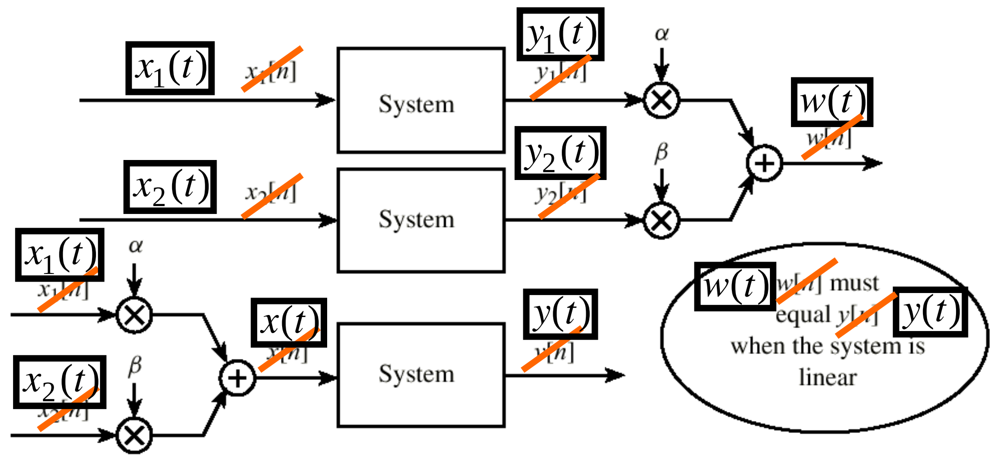
Linear and Time-Invariant (LTI) Systems

- If a continuous–time system is both linear and time–invariant, then the output ***y(t)*** is related to the input ***x(t)*** by a ***convolution integral***

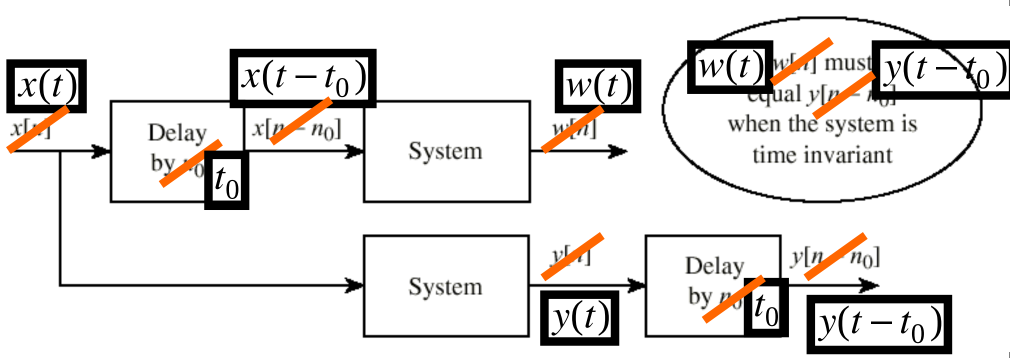
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

where ***h(t)*** is the ***impulse response*** of the system.

Testing for Linearity



Testing Time-Invariance



Ideal Delay: $y(t) = x(t - t_d)$

- **Linear**

$$ax_1(t - t_d) + bx_2(t - t_d) = ay_1(t) + by_2(t)$$

- **and Time–Invariant**

$$w(t) = x((t - t_0) - t_d)$$

$$y(t - t_0) = x((t - t_0) - t_d)$$

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

- **Linear**

$$\int_{-\infty}^{t} [ax_1(\tau) + bx_2(\tau)]d\tau = ay_1(t) + by_2(t)$$

- **And Time-Invariant**

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau \qquad \text{let } \sigma = \tau - t_0$$

$$\Rightarrow \quad w(t) = \int_{-\infty}^{t-t_0} x(\sigma)d\sigma = y(t - t_0)$$

Modulator: $y(t) = [A + x(t)]\cos\omega_c t$

- ***Not*** **linear--obvious because**

$$A + ax_1(t) + bx_2(t)] \neq$$
$$[A + ax_1(t)] + [A + bx_2(t)]$$

- ***Not*** **time-invariant**

$$w(t) = [A + x(t - t_0)]\cos\omega_c t \neq y(t - t_0)$$

Linear and Time-Invariant (LTI) Systems

- **If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a *convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$$

where *h(t)* is the *impulse response* of the system.

Convolution of Impulses, etc.

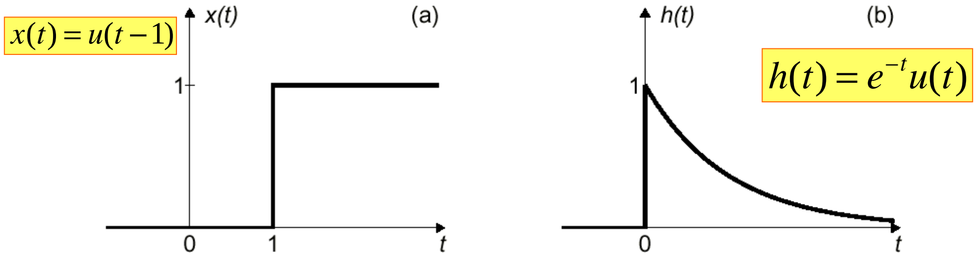
- **Convolution of two impulses**

$$\delta(t - t_1) * \delta(t - t_2) = \delta(t - t_1 - t_2)$$

- **Convolution of step and derivative of impulse**

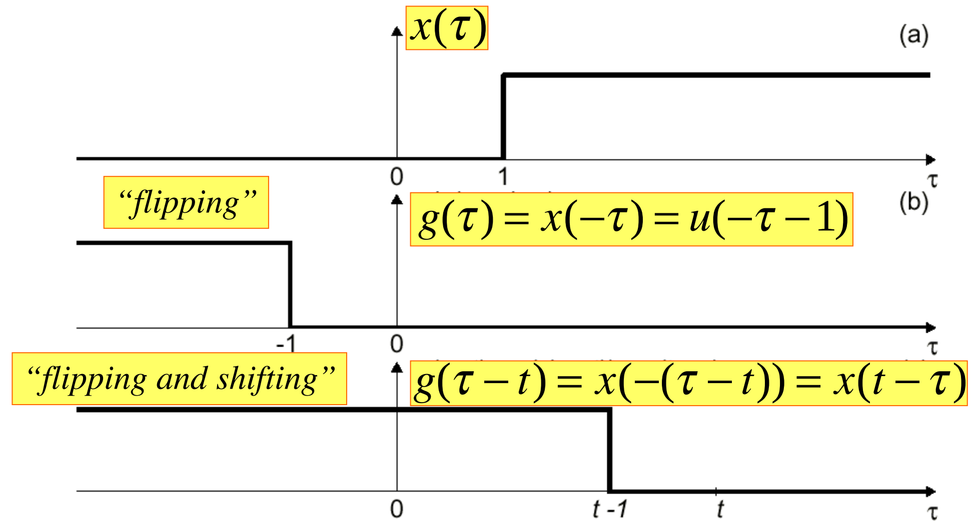
$$u(t) * \delta^{(1)}(t) = \delta(t)$$

Evaluating a Convolution

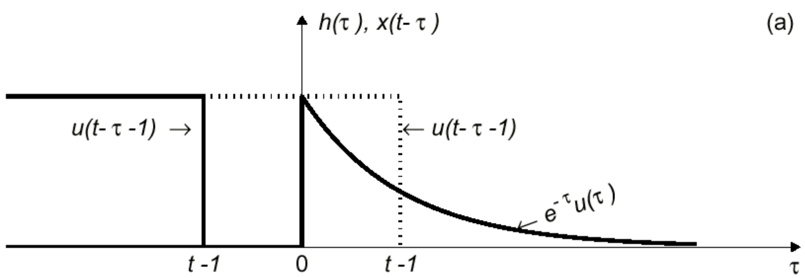


$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t)*x(t)$$

"Flipping and Shifting"



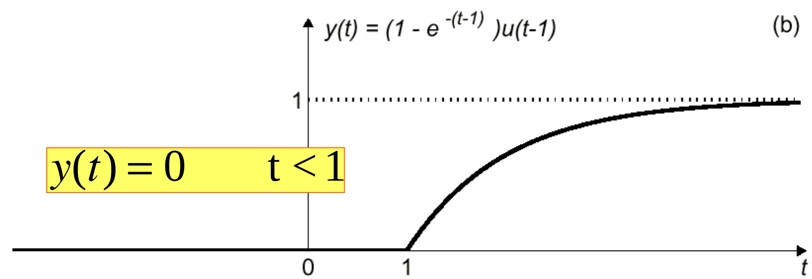
Evaluating the Integral



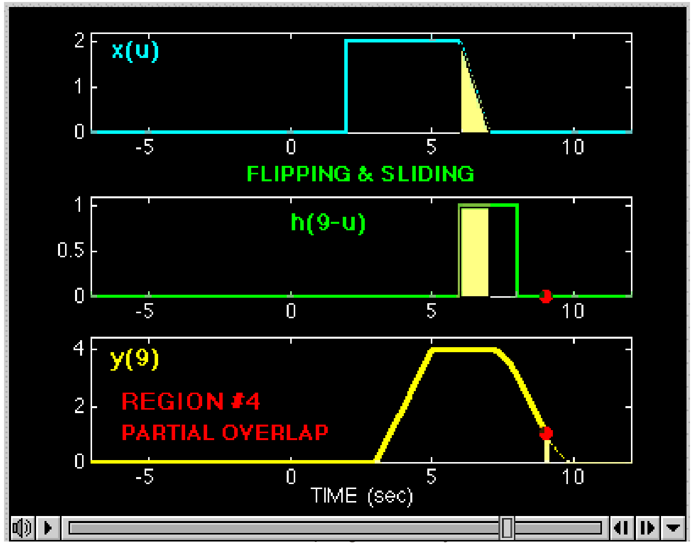
$$\begin{aligned} y(t) &= 0 && t-1<0 \\ &= \int_0^{t-1} e^{-\tau}d\tau && t-1\geq 0 \end{aligned}$$

Solution

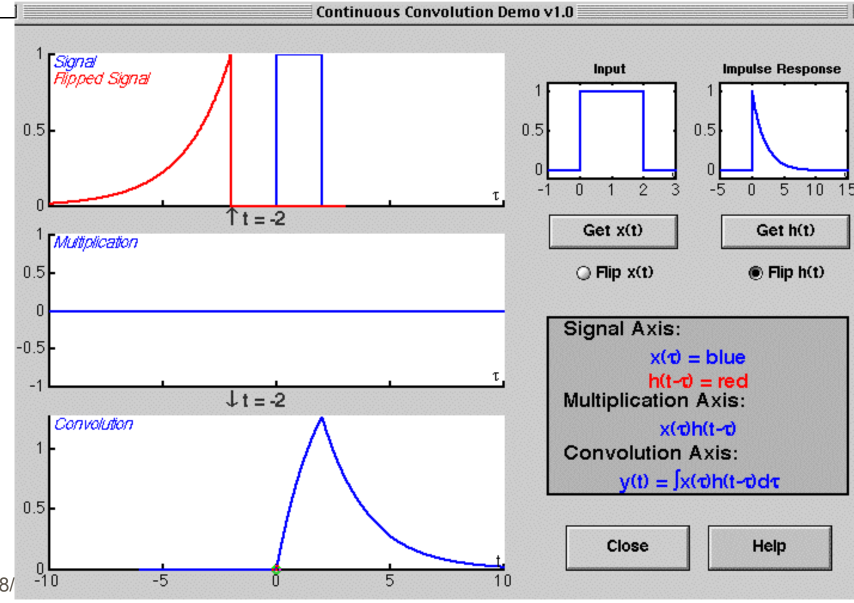
$$\begin{aligned} y(t) &= \int_0^{t-1} e^{-\tau}d\tau = -e^{-\tau}\Big|_0^{t-1} \\ &= 1-e^{-(t-1)} && t\geq 1 \end{aligned}$$



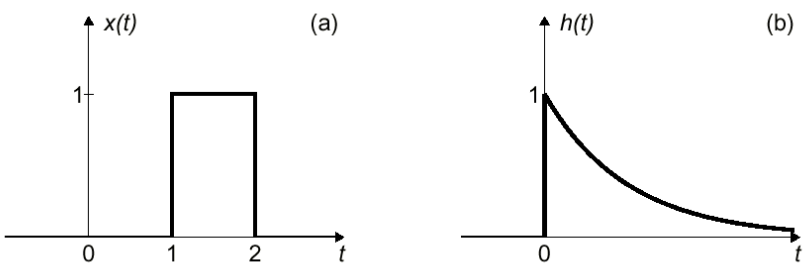
Convolution Demo



Convolution GUI

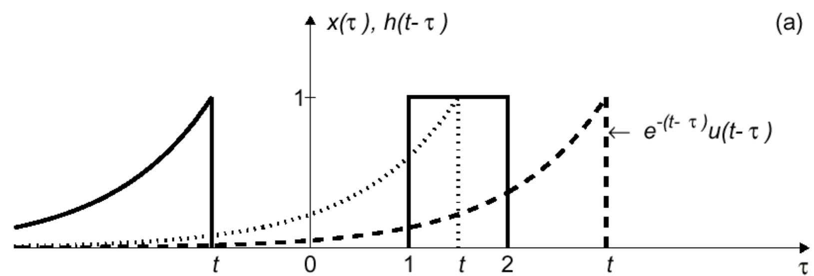


Another Convolution Example



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Evaluating the Integral

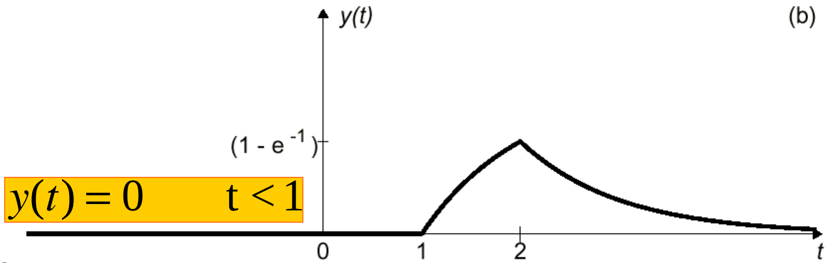


$$\begin{aligned} y(t) &= 0 & t < 1 \\ &= \int_{1}^{t} e^{-(t-\tau)} d\tau & 1 \le t \le 2 \\ &= \int_{1}^{2} e^{-(t-\tau)} d\tau & 2 \le t \end{aligned}$$

Solution

$$y(t) = \int_1^t e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^t = 1 - e^{-(t-1)} \quad 1 \le t \le 2$$

$$= \int_1^2 e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^2 = e^{-(t-2)} - e^{-(t-1)} \quad 2 \le t$$



Convolution is Commutative

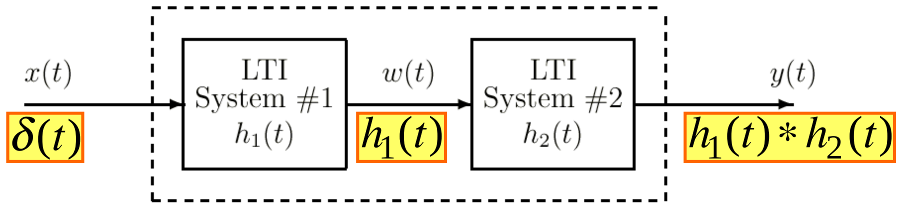
$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\text{let } \sigma = t - \tau \text{ and } d\sigma = -d\tau$$

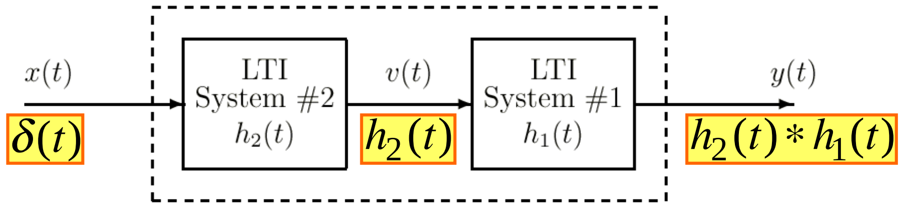
$$h(t) * x(t) = -\int_{\infty}^{-\infty} h(t-\sigma)x(\sigma)d\tau$$

$$= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\tau = x(t) * h(t)$$

Cascade of LTI Systems



$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$



(b)

Stability

- **A system is stable if every bounded input produces a bounded output.**
- **A continuous-time *LTI system* is stable if and only if**

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \le t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \quad \text{for} \quad t < 0$$

Convolution is Linear

- Substitute $x(t)=ax_1(t)+bx_2(t)$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [ax_1(\tau)+bx_2(\tau)]h(t-\tau)d\tau \\ &= a\int_{-\infty}^{\infty} x_1(\tau)h(t-\tau)d\tau + b\int_{-\infty}^{\infty} x_2(\tau)h(t-\tau)d\tau \\ &= ay_1(t)+by_2(t) \end{aligned}$$

Therefore convolution is linear.

Convolution is Time-Invariant

- Substitute $x(t-t_0)$

$$\begin{aligned} w(t) &= \int_{-\infty}^{\infty} h(\tau)x((t-\tau)-t_o)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x((t-t_o)-\tau)d\tau \\ &= y(t-t_o) \end{aligned}$$

Parallel LTI Systems