EE-2025 Spring-2000

Lecture 18

Introduction to the Fourier Transform

27-March-00

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 7-April
  - One page hand-written notes
  - Calculator
- Prob Set #9 is due this week
- Lab #9 is NUMERICAL FOURIER SERIES
  - New Functions: `fplot.m` and `quad8.m`

READING ASSIGNMENTS

- This Lecture:
  - Chapter 12, pp. 1200-1214
- Other Reading:
  - Recitation: Chapter 11
    - And Chapter 12, pp. 1207-1214, 1218-1222, 1232-1234
  - Next Lecture: Chapter 12, pp. 1214-1218, 1223-1229, and 1236-1241

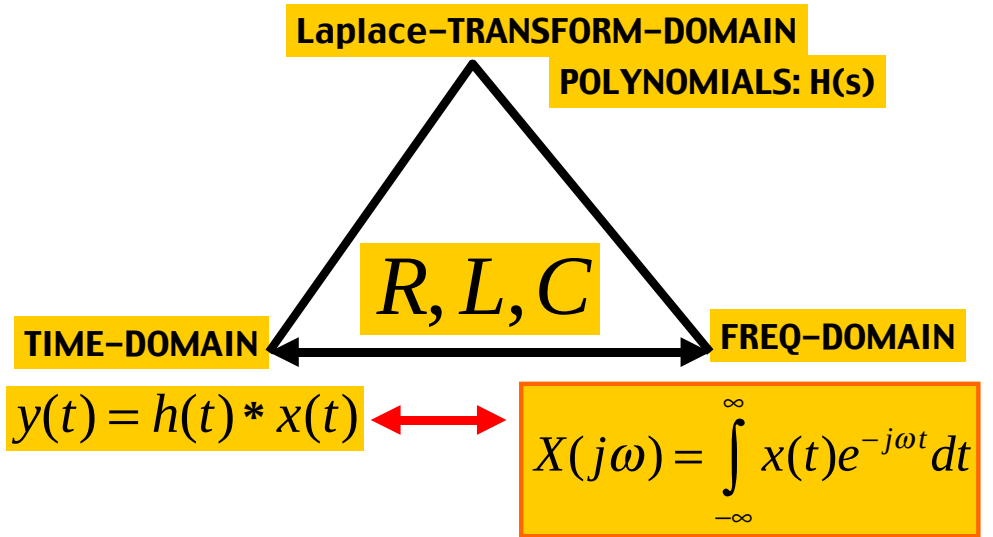
LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Relation to Fourier Series
- Examples of Fourier transform pairs

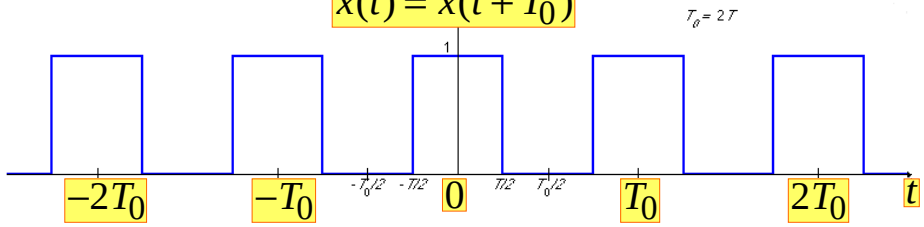
THREE DOMAINS: ANALOG



$$y(t) = h(t) * x(t) \quad \longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Series: Periodic *x(t)*

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

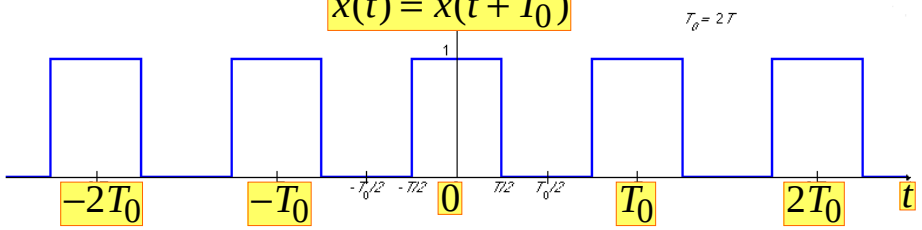
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal

$$x(t) = x(t + T_0)$$

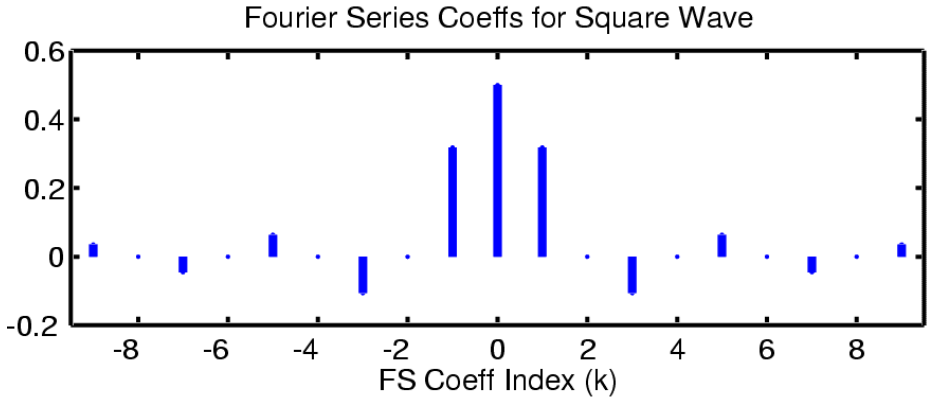


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Bigg|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

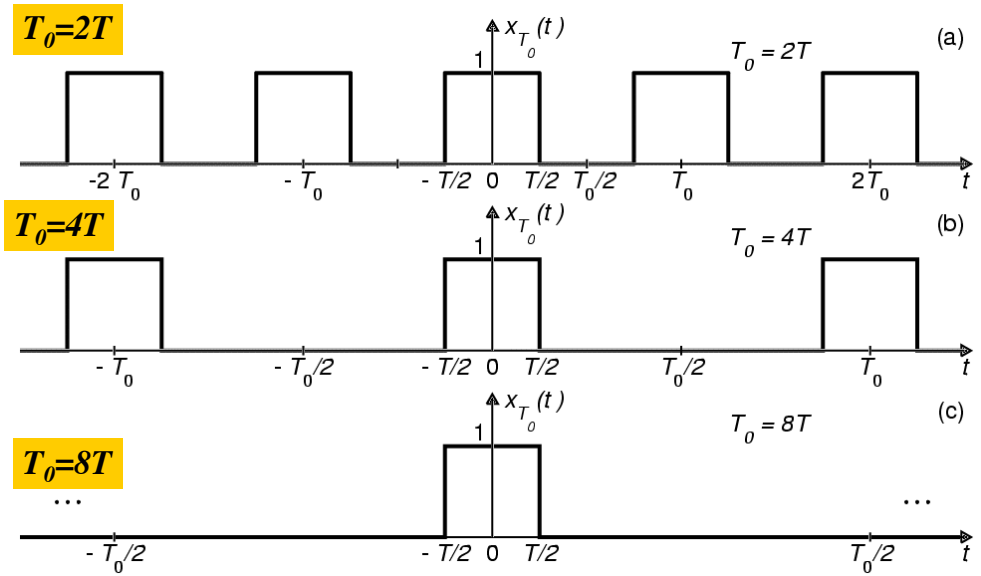
$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



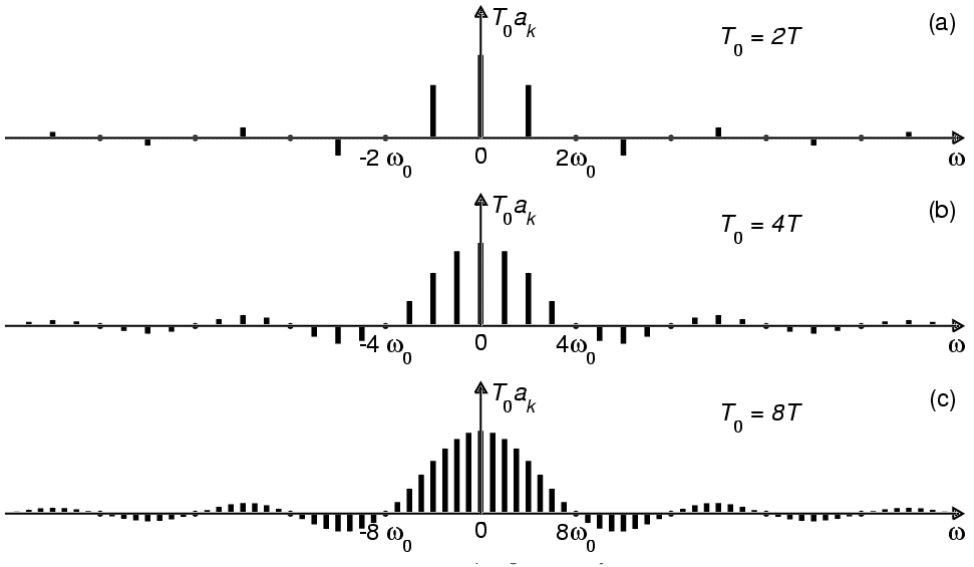
What if *x(t)* is not periodic?

- Sum of Sinusoids?
  - Non–harmonically related sinusoids
  - Would not be periodic, but would be non–zero for all *t*.
- Fourier transform
  - gives a "sum" (actually an integral) that involves ALL frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}(T_0 a_k)e^{jk\omega_0 t}\frac{2\pi}{T_0} \mapsto x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Synthesis

$$\lim_{T_0\to\infty}\frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0\to\infty}\frac{2\pi}{T_0}k = \omega$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-jk\omega_0 t}dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis

Fourier Transform

- For non-periodic signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$ ***Fourier Synthesis***

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$ ***Fourier Analysis***

Example 1: $x(t) = e^{-at}u(t)$

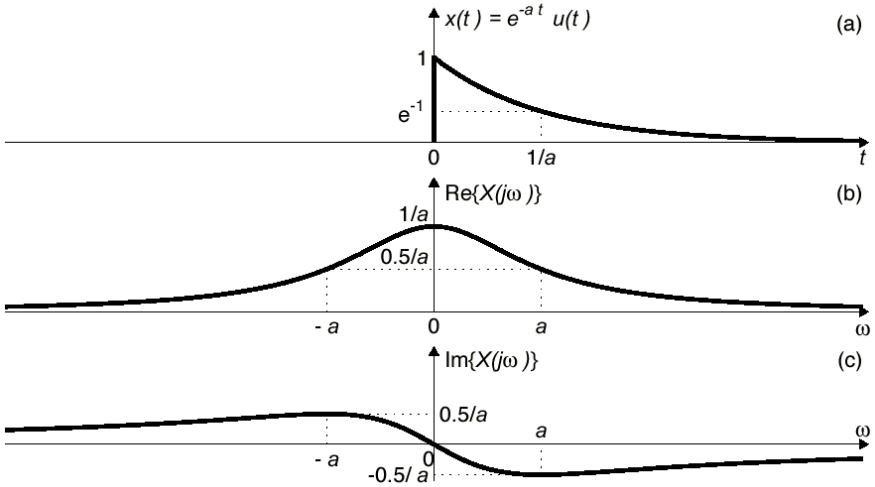
$$X(j\omega) = \int_0^{\infty} e^{-at}\, e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt$$

$a > 0$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\Bigg|_0^{\infty} = \frac{1}{a+j\omega}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

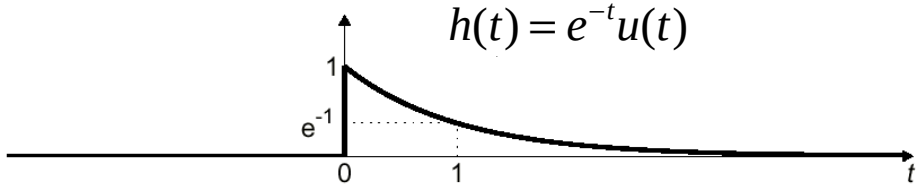
$$x(t) = e^{-at}u(t) \quad \Leftrightarrow \quad X(j\omega) = \frac{1}{a+j\omega}$$



Frequency Response

- Fourier Transform of h(t) is the Frequency Response

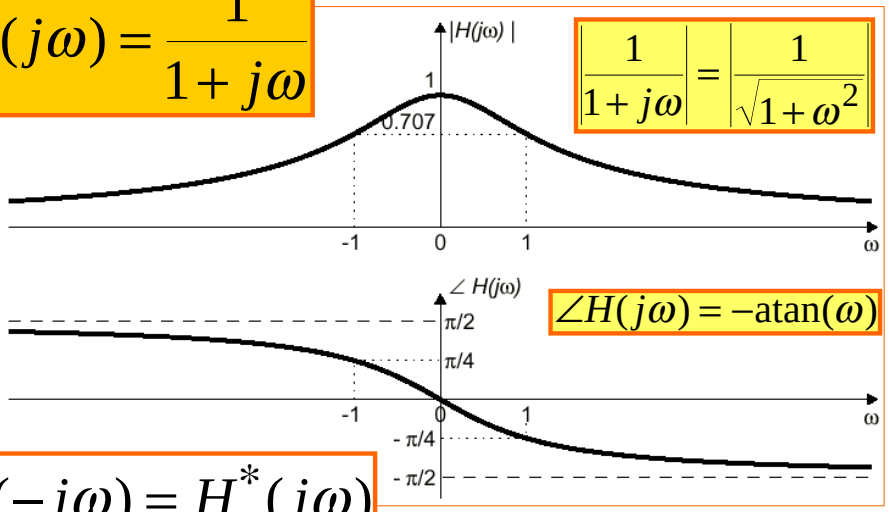
$$h(t) = e^{-t}u(t)$$



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1+j\omega}$$



$$\left|\frac{1}{1+j\omega}\right| = \left|\frac{1}{\sqrt{1+\omega^2}}\right|$$

$$\angle H(j\omega) = -\text{atan}(\omega)$$

$$H(-j\omega) = H^*(j\omega)$$

Example 2:

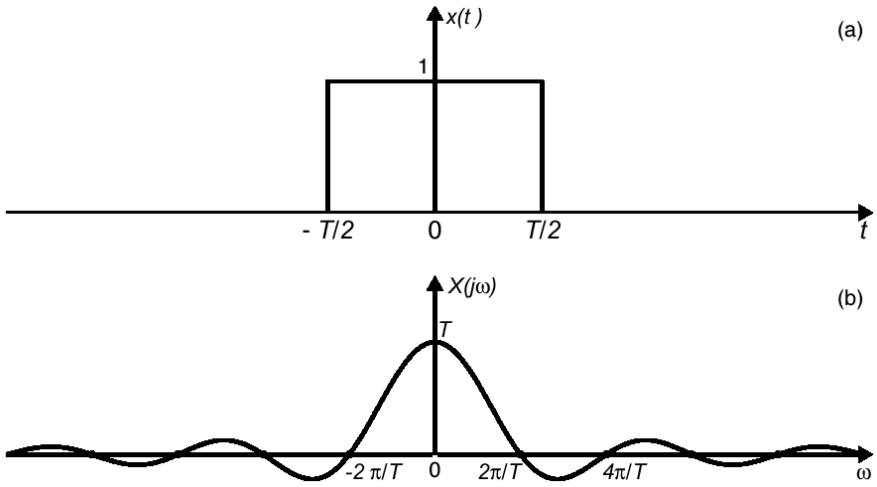
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t}dt = \int_{-T/2}^{T/2} e^{-j\omega t}dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3:

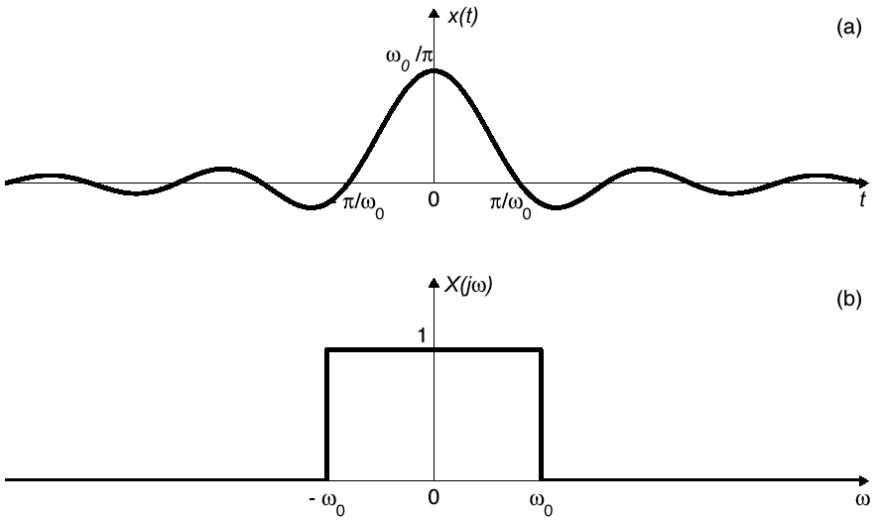
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi}\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t)=\frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega)=\begin{cases}1 & |\omega|<\omega_0\\ 0 & |\omega|>\omega_0\end{cases}$$



Example 4: $x(t)=\delta(t)$

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=1$$

Shifting Property of the Impulse

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t-t_0)e^{-j\omega t}dt=e^{-j\omega t_0}$$

$$x(t)=\delta(t)\Leftrightarrow X(j\omega)=1$$

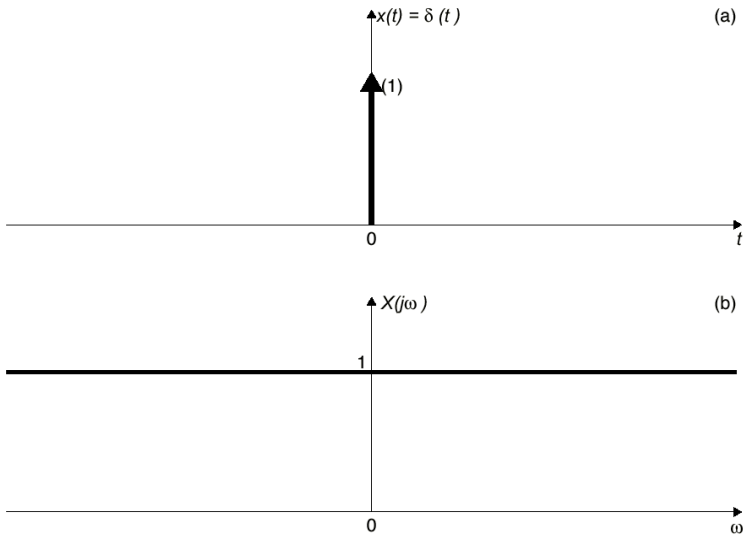


Table of Fourier Transforms

$$x(t)=e^{-at}u(t)\Leftrightarrow X(j\omega)=\frac{1}{a+j\omega}$$

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \quad \Leftrightarrow \quad X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t)=\frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega)=\begin{cases}1 & |\omega|<\omega_0\\ 0 & |\omega|>\omega_0\end{cases}$$

$$x(t)=\delta(t-t_0)\Leftrightarrow X(j\omega)=e^{-j\omega t_0}$$