EE-2025 Spring-2000

Lecture 19

Fourier Transform Properties

31-March-00

Info: Web-CT, Lab, HW

- Calendar: Quiz #3 is 7–April
  - REVIEW on THURSDAY at 7PM
- Prob Set #10 is due NEXT WEEK
  - Solution will be posted Thursday @ 6pm
- Labs
  - #10: Fourier–Based Design
  - #11: Communication/AM
  - #12: Poles & Zeros (PeZ)

READING ASSIGNMENTS

- This Lecture:
  - Chapter 12, pp. 1214–1218, 1223–1229, and 1236–1241
- Other Reading:
  - Recitation: All of Chapter 12
  - Next Lecture: Chapter 13, pp. 1317–1331

LECTURE OBJECTIVES

- The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - Convolution property
  - Multiplication property

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis (Forward Transform)

$$\text{Time-domain} \Leftrightarrow \text{Frequency-domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- **Manipulate the "Frequency Spectrum"**
- **Analog Communication Systems**
  - **AM: Amplitude Modulation; FM**
  - **What are the "Building Blocks" ?**
    - **Abstract Layer, not implementation**
- **Ideal Filters: mostly BPFs**
- **Frequency Shifters**
  - **Modulators, or Multipliers: x(t)p(t)**

Strategy for using the FT

- **Develop a set of known Fourier transform pairs.**
- **Develop a set of "theorems" or properties of the Fourier transform.**
- **Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.**

Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

Example 4: $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

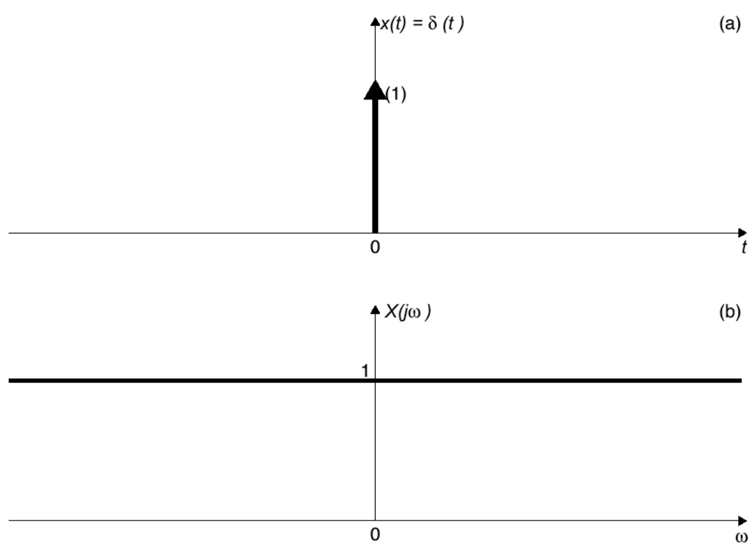
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

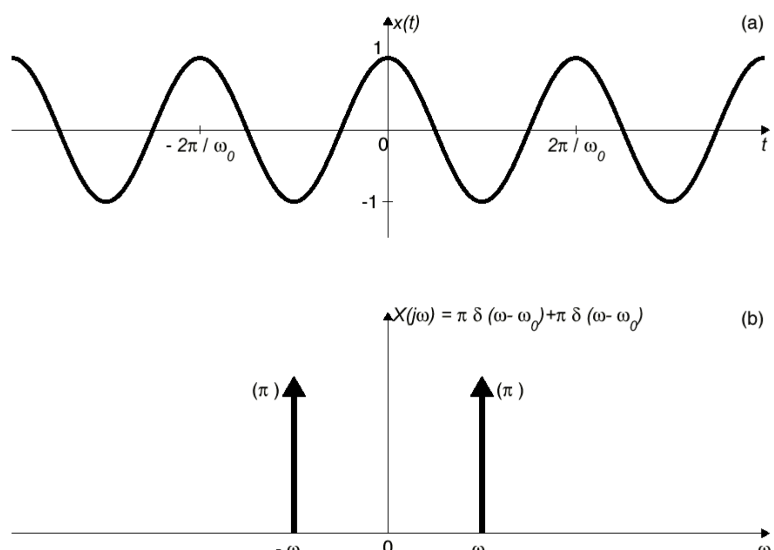
$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$



$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform of a General Periodic Signal

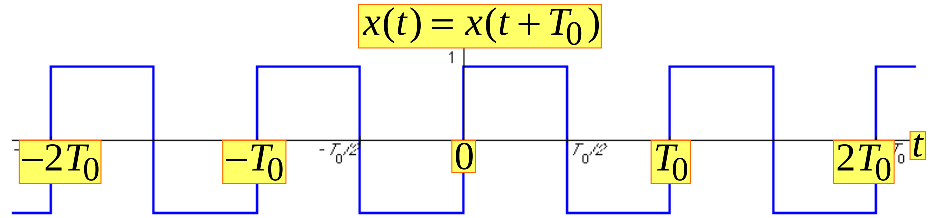
- If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

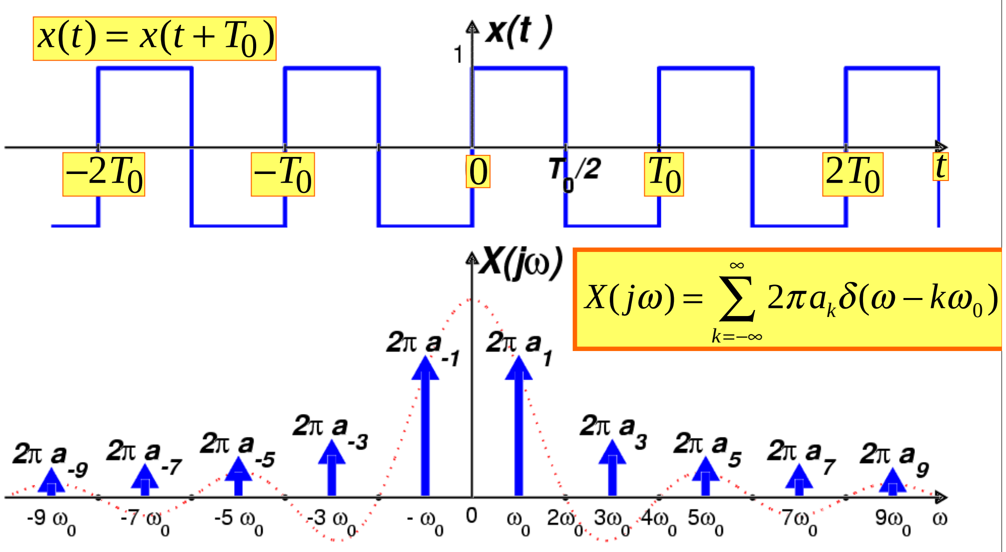
Square Wave Signal



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1) e^{-j\omega_0 kt}\, dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 kt}\, dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

Square Wave SPECTRUM



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Table of Easy FT Properties

Linearity Property

$$a x_1(t) + b x_2(t) \Leftrightarrow a X_1(j\omega) + b X_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

$$\int_{-\infty}^{\infty} x(t-t_d)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_d)}d\tau$$

$$= e^{-j\omega t_d} X(j\omega)$$

For example, $e^{-a(t-5)}u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a+j\omega}$

Scaling Property

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

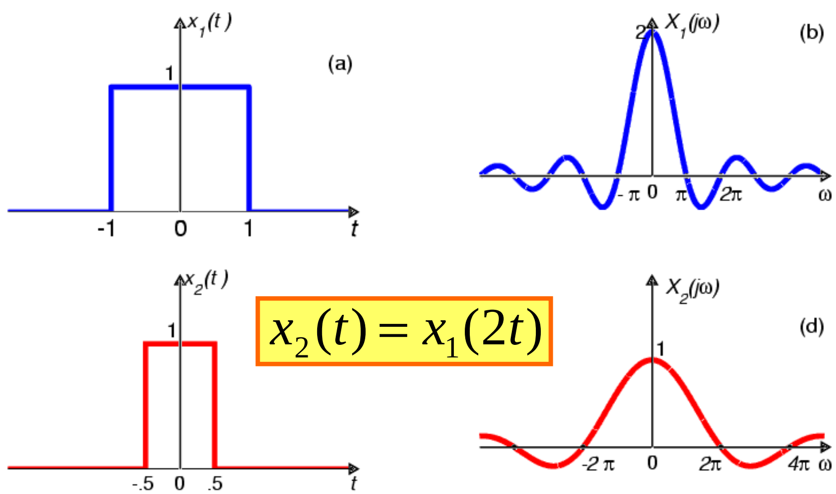
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau/a)}(\tfrac{1}{|a|})d\tau$$

$$= \tfrac{1}{|a|} X(j\tfrac{\omega}{a})$$

For example, $e^{-at}u(t) \Leftrightarrow \dfrac{1}{a}\dfrac{1}{(1+j\frac{\omega}{a})} = \dfrac{1}{a+j\omega}$

Scaling Property

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j(\tfrac{\omega}{a}))$$



$$x_2(t) = x_1(2t)$$

Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- Cannot simultaneously reduce time duration and bandwidth

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

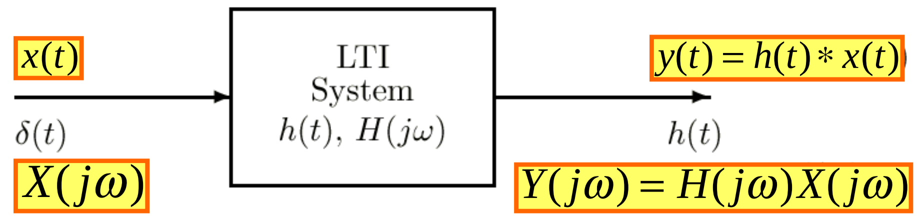
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- **Convolution in the time-domain**

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

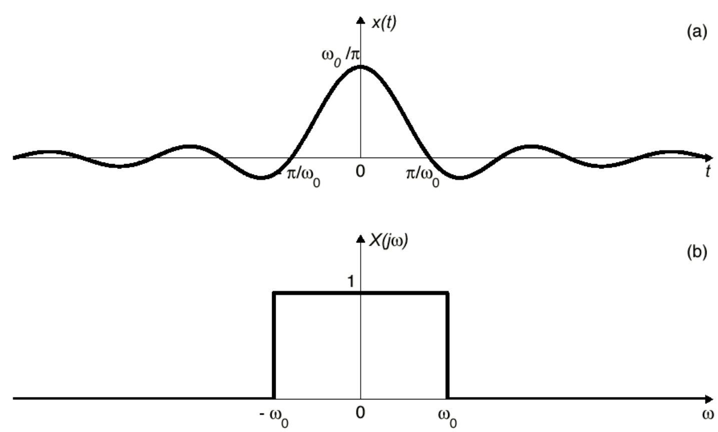
corresponds to MULTIPLICATION in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Convolution Example

- **Bandlimited Input Signal**
  - **"sinc" function**
- **Ideal LPF (Lowpass Filter)**
  - **h(t) is a "sinc"**
- **Output is Bandlimited**
  - **Convolve "sincs"**

Ideally Bandlimited Signal

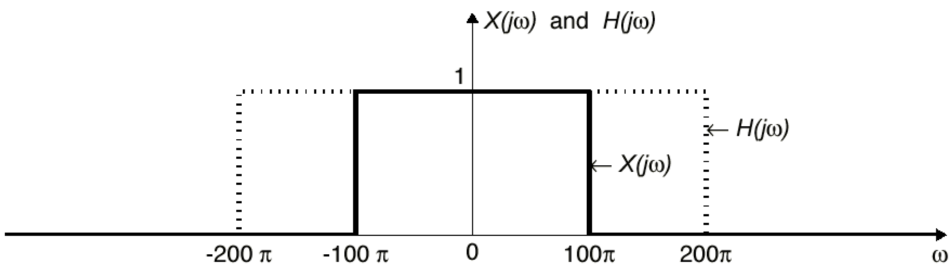
$$x(t) = \frac{\sin(100\pi t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

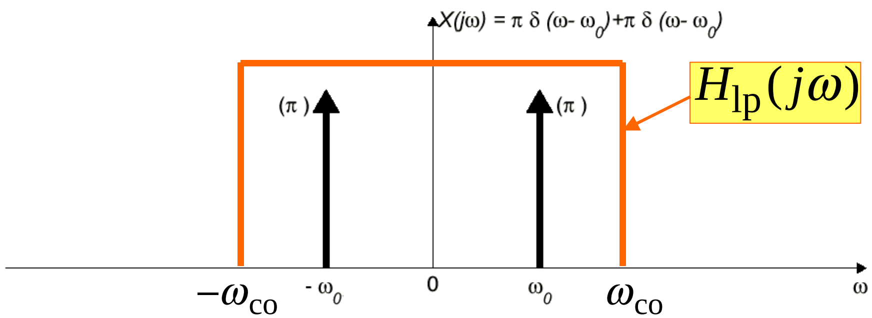
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega-\omega_0) + H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

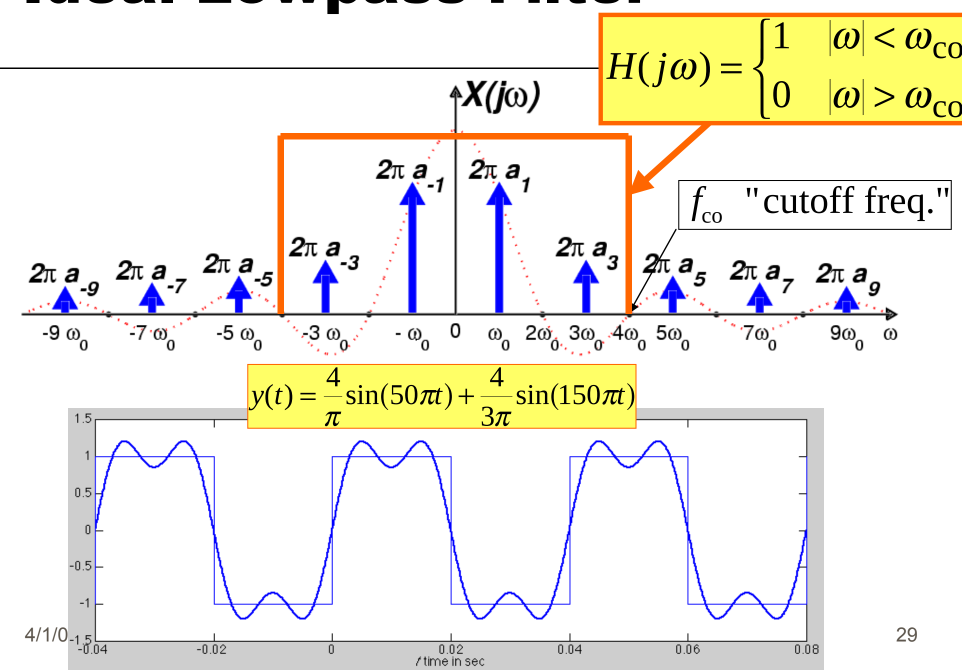
Ideal Lowpass Filter



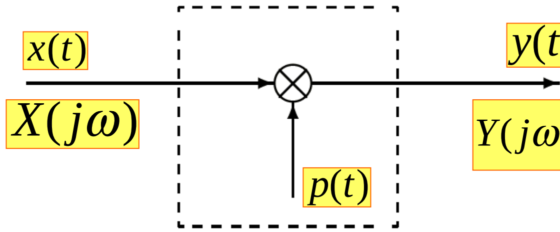
$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter



Signal Multiplier (Modulator)



$y(t) = p(t)x(t)$

$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

- **Multiplication in the time-domain corresponds to convolution in the frequency-domain.**

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{\infty} e^{-j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(j(\omega-\omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega-\omega_0)) + \frac{1}{2} X(j(\omega+\omega_0))$$