EE-2025 Spring-2000

Lecture 21

Sampling and Reconstruction

10-April-00

Info: Web-CT, Lab, HW

- Calendar: Final Exam
  - Period 5, Tues, 2–May
  - Review on Monday nite, 1–May
- Reading Assignment: Ch 13 of Notes.
- Prob Set #11 – due this week
- Lab #10 due week of 17–April
  - Lab #11 due week of 24–April
  - ALL Labs must be turned in by 28–April

LECTURE OBJECTIVES

- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- Review of FT properties
  - Convolution <––> multiplication
  - Frequency shifting
  - Review of AM

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

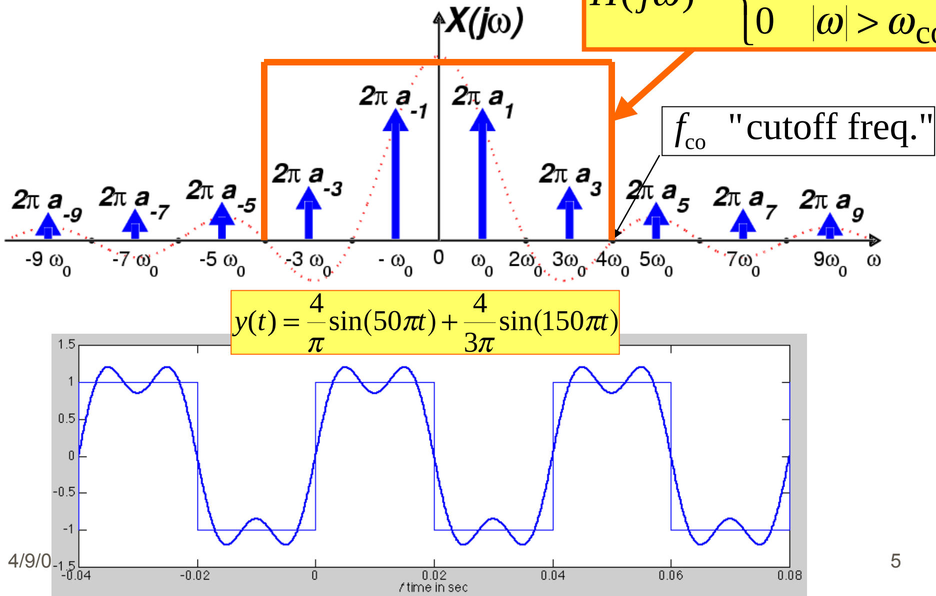
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

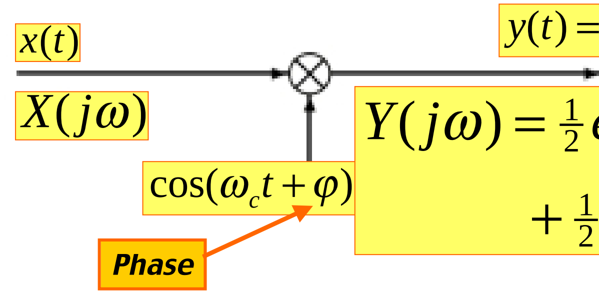
Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

Amplitude Modulator



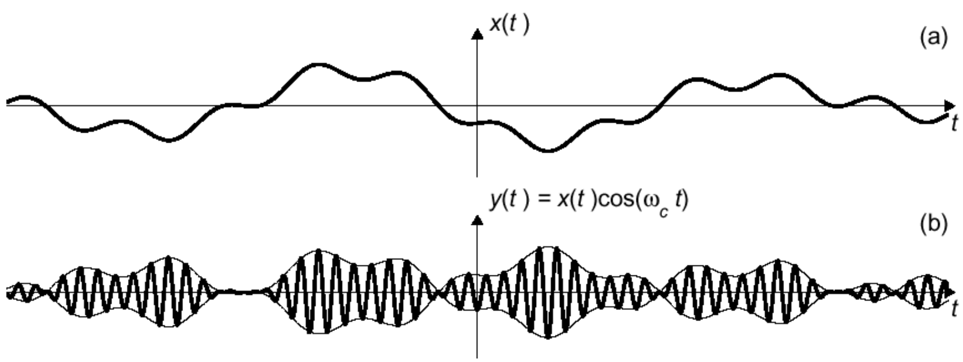
$$y(t) = x(t)\cos(\omega_c t + \varphi)$$

$$Y(j\omega) = \tfrac{1}{2}e^{j\varphi}X(j(\omega-\omega_c)) + \tfrac{1}{2}e^{-j\varphi}X(j(\omega+\omega_c))$$

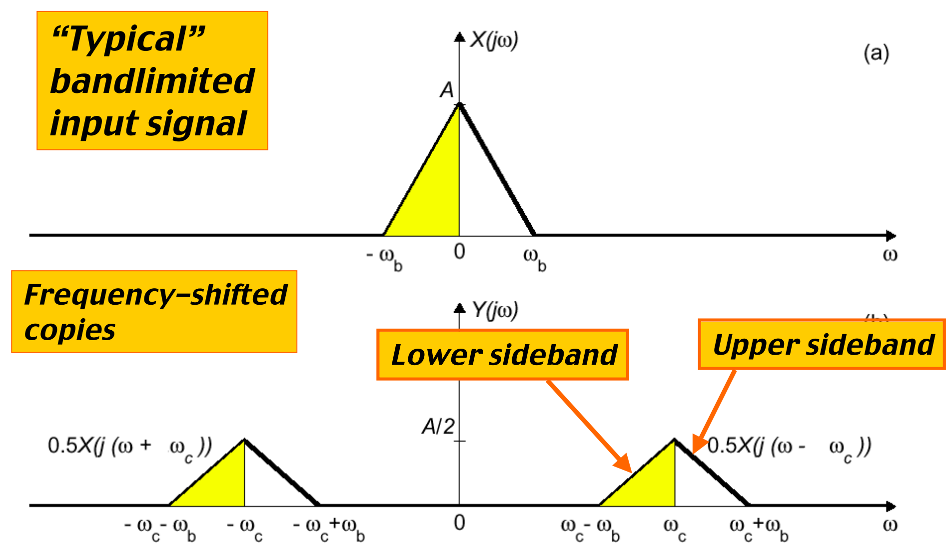
- ***x(t)*** **modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of *X(jω)*.**

Double Sideband AM (Time-Domain)

- **In the time-domain, the envelope of sinewave peaks follows |*x(t)*|**

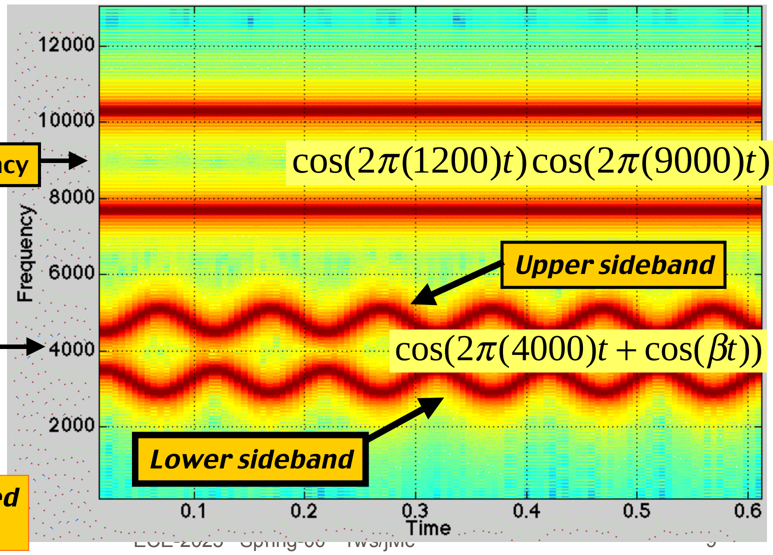


DSBAM: Frequency-Domain



DSBAM: Frequency-Domain

Sinusoid & Chirp are bandlimited input signals



Carrier Frequency → $\cos(2\pi(1200)t)\cos(2\pi(9000)t)$

Upper sideband

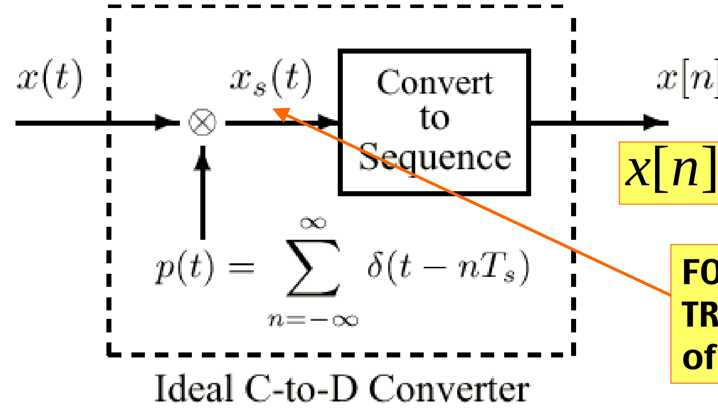
Carrier Frequency → $\cos(2\pi(4000)t+\cos(\beta t))$

Lower sideband

Frequency-shifted copies

Ideal C-to-D Converter

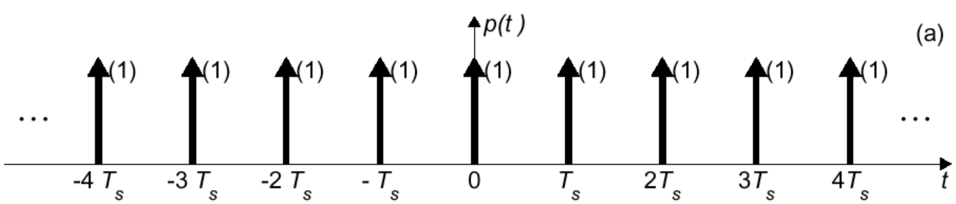
- **Mathematical Model for A-to-D**



$$x[n]=x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train

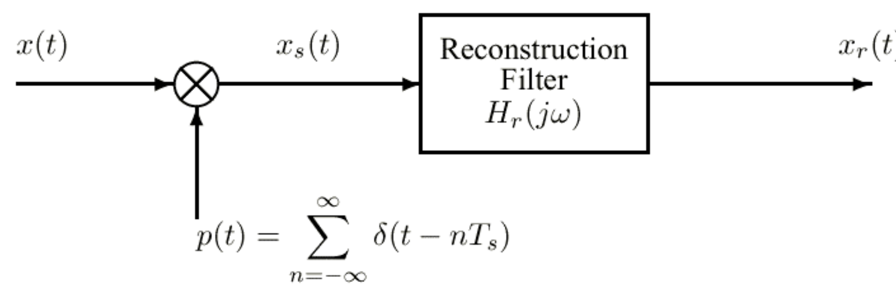


$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_s t}$$

$$a_k=\frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-jk\omega_s t}dt=\frac{1}{T_s}$$

$$\omega_s=\frac{2\pi}{T_s}$$

Impulse Train Sampling

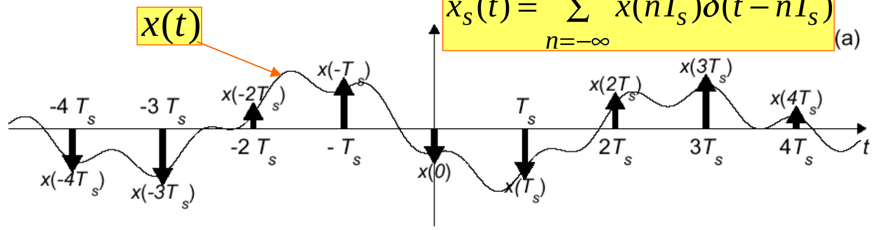


$$x_s(t)=x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{n=-\infty}^{\infty}x(t)\delta(t-nT_s)$$

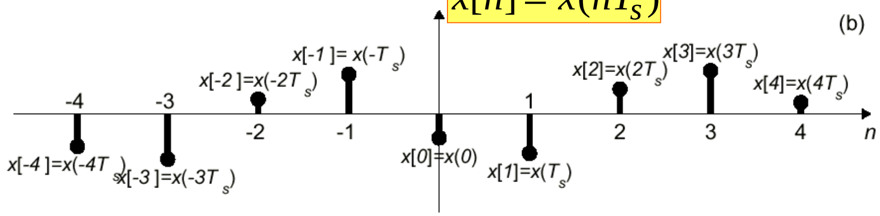
$$x_s(t)=\sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$

Illustration of Sampling

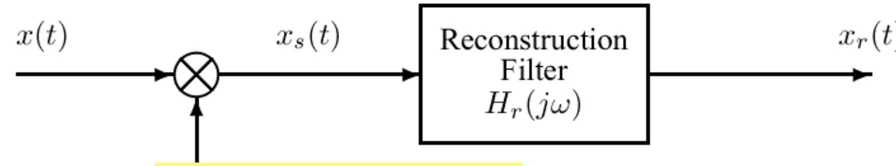
$$x_s(t)=\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$



$$x[n]=x(nT_s)$$



Sampling: Freq. Domain



$$p(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t)=x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

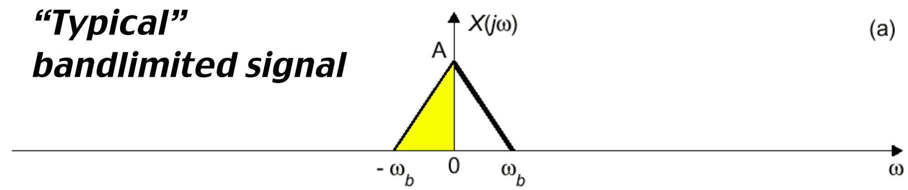
$$x_s(t)=x(t)\sum_{k=-\infty}^{\infty}\frac{1}{T_s}e^{jk\omega_s t}=\frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

$$X_s(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

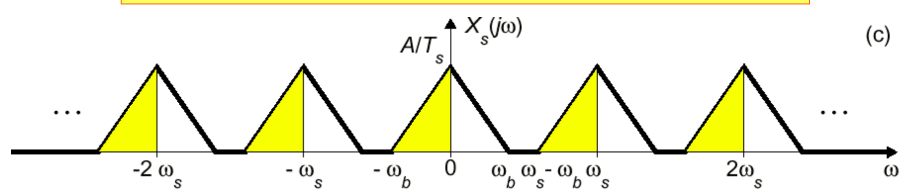
$$\omega_s=\frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

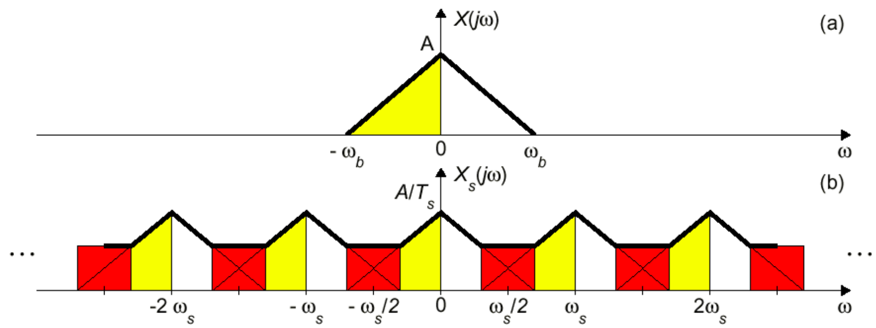


$$X_s(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

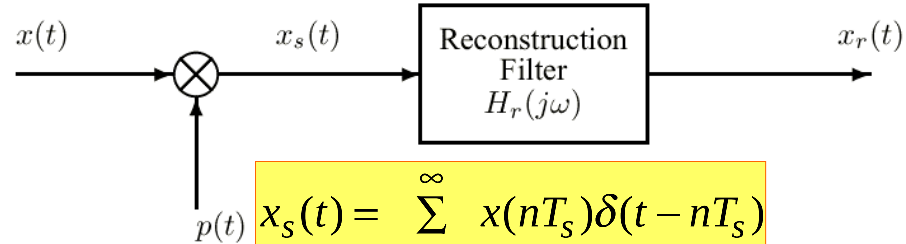


Aliasing Distortion

- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

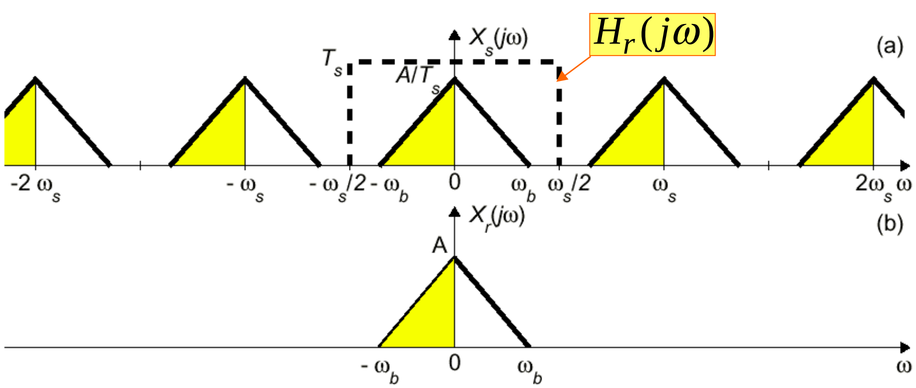


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

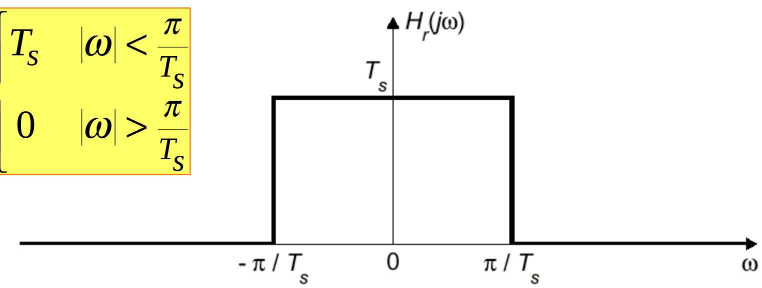
Reconstruction in the Frequency-Domain



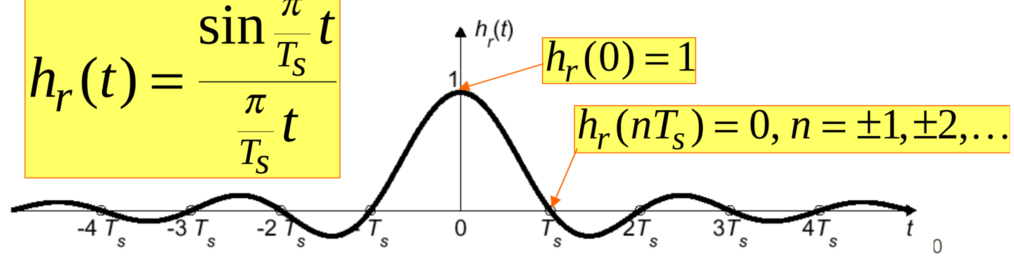
- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)\,X_s(j\omega)$.

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \dfrac{\pi}{T_s} \\ 0 & |\omega| > \dfrac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

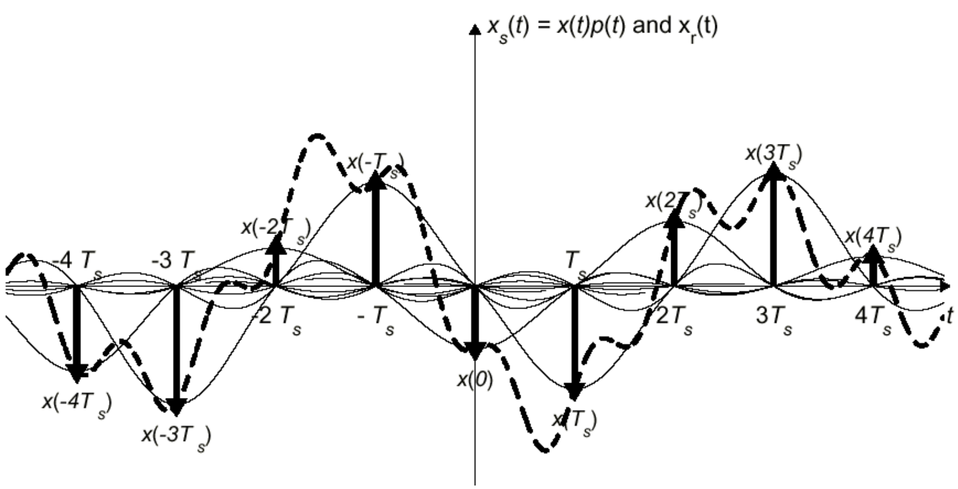
Shannon Sampling Theorem

- **"SINC" Interpolation is the ideal**
  - **PERFECT RECONSTRUCTION**
  - **of BANDLIMITED SIGNALS**

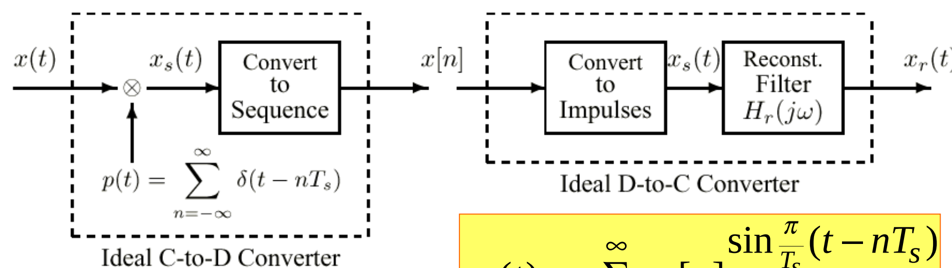
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$