EE-2025 Spring-2000

Lecture 23

IIR Filters: Feedback

17-April-00

Info: Web-CT, Lab, HW

- Final Exam, 2–May @ 11:30 AM
  - Calculator, 1 page Handwritten Notes
  - Review: Monday, 1–May @ 6 pm
- Prob Set #12 is due this week
  - HW #13 due last day (in Lecture)
- Help Sessions
  - Mon (6:00) & Tues (6:00) in VL–456
  - TAs and Profs have office hours

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 249–263
- Other Reading:
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, pp. 269–282

LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define IIR DIGITAL Filters
  - Have FEEDBACK: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

  - Show how to compute the output y[n]
    - FIRST–ORDER CASE (N=1)
    - Z–transform: Impulse Response h[n] <––> H(z)

L-pt RUNNING AVG h[n]

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE

$$h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$$

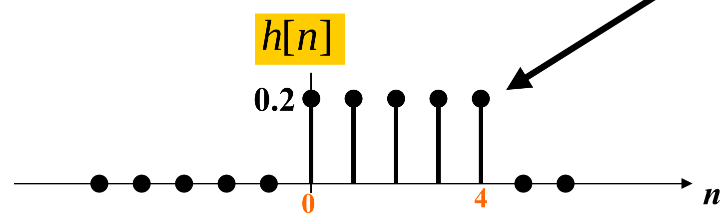
STEP RESPONSE

$$x[n] = u[n] \quad \Rightarrow \quad y[n] = ?$$

MATH FORMULA for h[n]

- Use **SHIFTED** IMPULSES to write h[n]

$$h[n] = \tfrac{1}{5}\delta[n] + \tfrac{1}{5}\delta[n-1] + \tfrac{1}{5}\delta[n-2] + \tfrac{1}{5}\delta[n-3] + \tfrac{1}{5}\delta[n-4]$$



$$\{b_k\} = \{\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\}$$

LTI: Convolution

- Output = Convolution of x[n] & h[n]
  - NOTATION: y[n] = x[n]*h[n]
  - Here is the FIR case:

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Same as b_k

FINITE LIMITS

CONVOLUTION Example

$$h[n] = \tfrac{1}{5}\delta[n] + \tfrac{1}{5}\delta[n-1] + \tfrac{1}{5}\delta[n-2] + \tfrac{1}{5}\delta[n-3] + \tfrac{1}{5}\delta[n-4]$$

$$x[n] = u[n]$$

n	-1	0	1	2	3	4	5	6	7
$x[n]$	0	1	1	1	1	1	1	1	...
$h[n]$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0	0
	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
	0	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$y[n]$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	1	1	...

L-pt RUNNING AVG: Step Response

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

STEP RESPONSE

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} u[n-k] = \begin{cases} \frac{n+1}{L} & n = 0,1,2,\ldots L-1 \\ 1 & n \geq L \\ 0 & n < 0 \end{cases}$$

L-pt RUNNING AVG H(z)

$$H(z) = \sum_{k=0}^{L-1} \tfrac{1}{L} z^{-k} = \frac{1-z^{-L}}{L(1-z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z-1)}$$

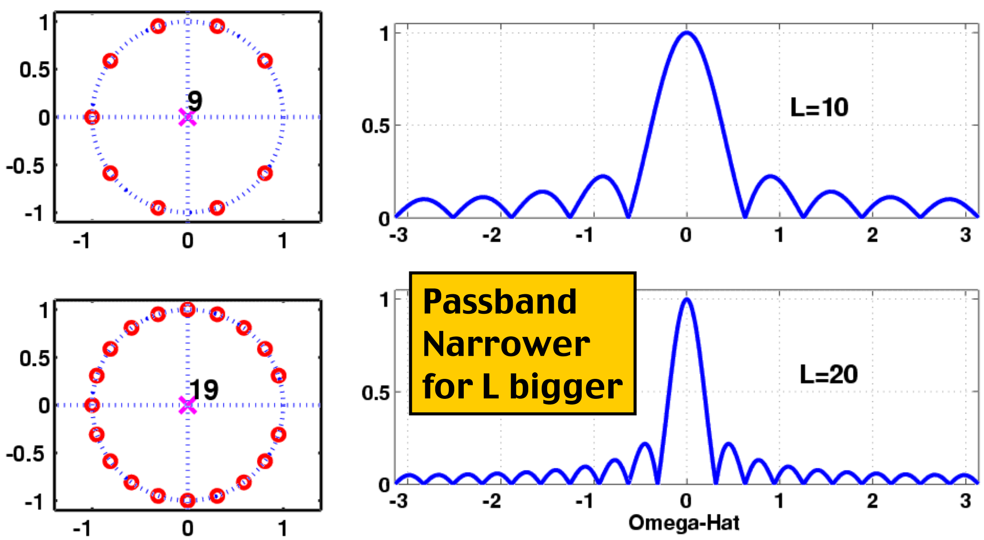
$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1,2,\ldots L\text{-}1$$

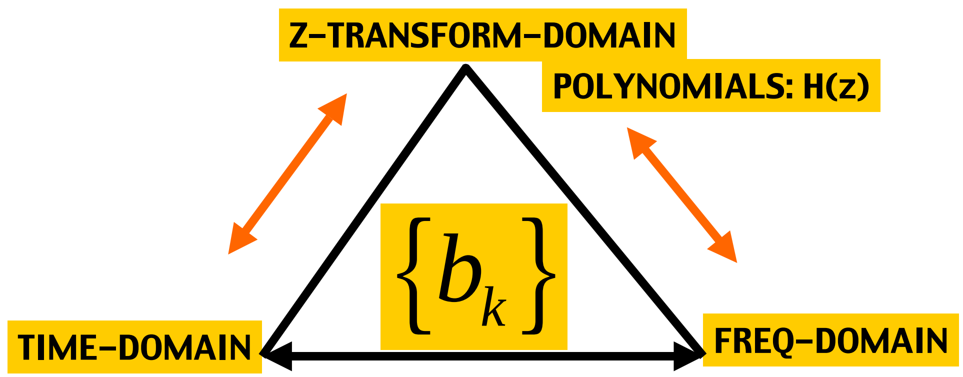
ZEROS on UNIT CIRCLE

(z−1) in denominator cancels k=0 term

FILTER DESIGN: CHANGE L



THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define IIR Filters
  - Have FEEDBACK: PREVIOUS OUTPUTS

$$y[n]=\sum_{\ell=1}^{N} a_\ell y[n-\ell]+\sum_{k=0}^{M} b_k x[n-k]$$

- Show how to compute the output y[n]
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response h[n] <--> H(z)

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, h[n]
  - INFINITELY LONG
  - IIR Filters

$$H(z)=\sum_{n=0}^{\infty} h[n]z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n]=a_1 y[n-1]+b_0 x[n]+b_1 x[n-1]$$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n]=0.8y[n-1]+3x[n]-2x[n-1]$$

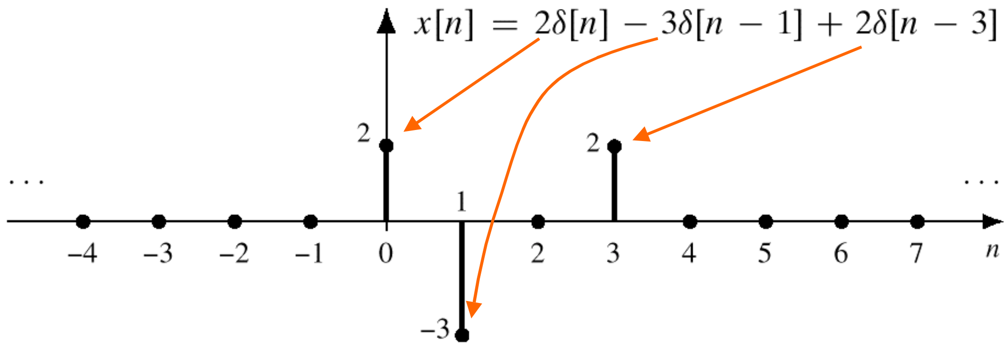
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$



COMPUTE y[n]

- **FEEDBACK DIFFERENCE EQUATION:**

$$y[n] = 0.8y[n-1] + 5x[n]$$

- **NEED y[−1] to get started**

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- **y[n] = 0, for n<0**
- **BECAUSE x[n] = 0, for n<0**

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- **THIS STARTS THE RECURSION:**

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- **SAME with MORE FEEDBACK TERMS**

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

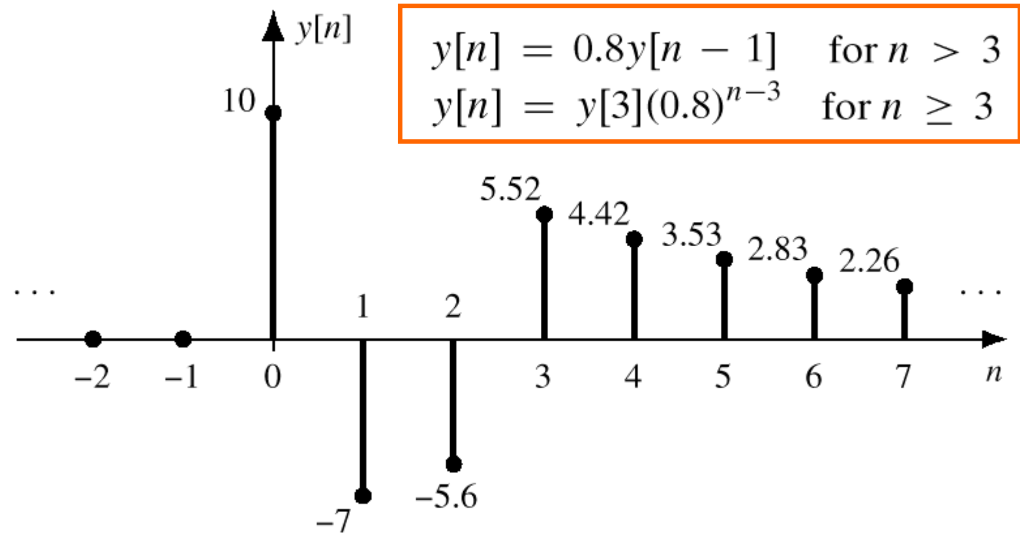
COMPUTE MORE y[n]

- **CONTINUE THE RECURSION:**

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$
$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$
$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE

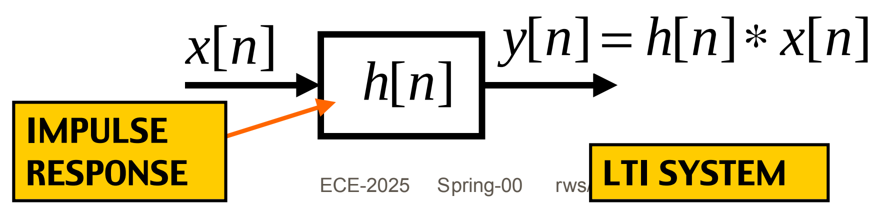
- **DIFFERENCE EQUATION:**

$$y[n] = 0.8y[n-1] + 3x[n]$$

- **Find h[n]**

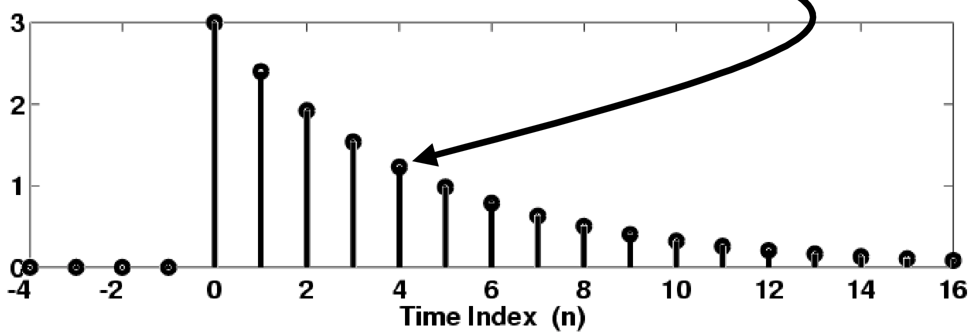
$$h[n] = 3(0.8)^n u[n]$$

- **CONVOLUTION in TIME-DOMAIN**



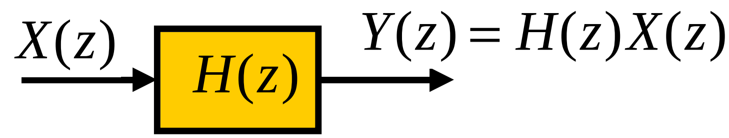
PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$

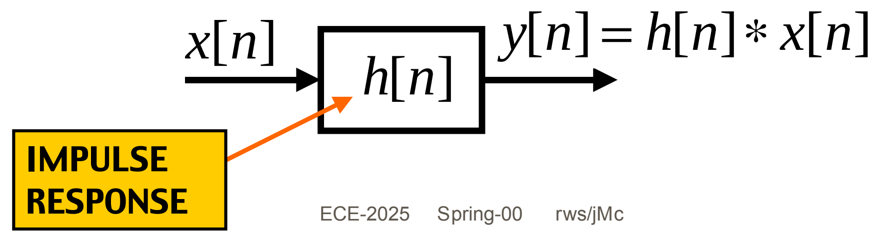


CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



Infinite-Length Signal: h[n]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- COMPACT FORM for H(z)

$$H(z) = \sum_{n=0}^{\infty} b a^n z^{-n} = \sum_{n=0}^{\infty} b (a z^{-1})^n$$

$$= \frac{b}{1 - a z^{-1}} \quad \text{if } |z| > |a|$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 a^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

H(z) = z-Transform{ h[n] }

- **FIRST–ORDER IIR FILTER:**

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0\, a^n u[n] + b_1\, a^{n-1} u[n-1]$$

$$H(z) = \frac{b_0}{1-a_1 z^{-1}} + \frac{b_1 z^{-1}}{1-a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n] = 1, \quad \text{for } n \geq 0$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
.		.

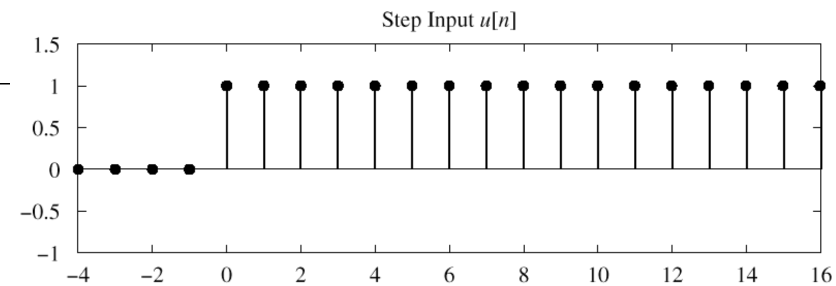
DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \quad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE



$y[n] = 0.8y[n-1] + 5x[n]$